1 Ongoing Research

1.1 Motivation: Affine Bezout-type theorems

The underlying motivation for my Ph.D. thesis [18] and recent works (e.g. [15], [16], [17]) is to understand 'affine Bezout-type' theorems. The usual version of Bezout theorem gives a formula for the number of solutions (counted with multiplicities) of n polynomials on the *projective* space $\mathbb{P}^n(\mathbb{C})$ as the product of the degrees of these polynomials. On the contrary, affine Bezout-type theorems estimate the number of solutions of a system of equations on an *affine* variety. In [17] (which is an outgrowth of a part of my Ph.D. thesis), I introduce and study projective completions of affine algebraic varieties determined by 'degree-like functions'. Currently I am working on a continuation (also based on my thesis) of [17] in which I present a general framework for affine Bezout-type theorems based on the theory developed in [17]. In particular, in the Ph.D. thesis I introduced a special class of projective completions that do not 'add solutions at infinity' and which are determined by 'degree-like functions'. It follows from the 'normalization' of degree-like functions (introduced in [17]) that for the purpose of adding no solutions at infinity, it suffices to consider only 'subdegrees'. But we must restrict the degree-like functions to an even smaller subclass in order to achieve desired formulae for estimating the size of a generic fiber of a finite-to-one polynomial mapping. With that in mind, I studied in my Ph.D. thesis a special class of semidegrees called *iterated semidegrees*. In the forthcoming article [17] I show that the degree of a completion (of an affine variety) corresponding to an iterated semidegree can be explicitly calculated and this in turn implies that the number of solutions of a system of equations *preserved* by an iterated semidegree can also be explicitly computed. My hope is that subdegrees corresponding only iterated semidegrees are sufficient for affine Bezout-type theorems and this would lead to computable affine Bezout-type theorems.

1.2 Projective completion of affine varieties via 'degree-like functions'

In [17] and my thesis I introduce and study a class of projective completions of affine algebraic varieties which generalize the construction of toric varieties from convex rational polytopes.

A familiar construction of completions of affine algebraic varieties is via *filtrations* on their coordinate rings: let X be an arbitrary affine variety over an algebraically closed field $\mathbb{K}$ and $\mathcal{F} = \{F_d : d \geq 0\}$ be a filtration on the coordinate ring $\mathbb{K}[X]$ of X (which in our case means that $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ is a sequence of vector subspaces of $\mathbb{K}[X]$ such that $\mathbb{K}[X] = \bigcup_{d \geq 0} F_d$, and $F_d F_e \subseteq F_{d+e}$). Then (under certain natural assumptions) $\operatorname{Proj} \bigoplus_{d \geq 0} F_d$ is a *completion* of X, i.e. a complete variety which contains X as a dense open subset (cf. [17, Proposition 2.2]).

Giving a filtration on $\mathbb{K}[X]$, on the other hand, is equivalent to defining a *degree-like function* $\delta : \mathbb{K}[X] \to \mathbb{Z}$ which satisfies the following properties:

1. $\delta(f+g) \leq \max\{\delta(f), \delta(g)\}$ for all $f, g \in \mathbb{K}[X]$, with $<$ in the preceding equation implying $\delta(f) = \delta(g)$, and
2. $\delta(fg) \leq \delta(f) + \delta(g)$ for all $f, g \in \mathbb{K}[X]$.

The vector spaces $F_d := \{f \in \mathbb{K}[X] : \delta(f) \leq d\}$ define a filtration on $\mathbb{K}[X]$ associated with δ and from that filtration, one constructs a completion $\bar{X}^{\delta}$ of X as in the first paragraph. For $X = \mathbb{K}^n$ and δ equal to the usual degree of polynomials, the completion of $\mathbb{K}^n$ we get via this construction is the *standard projective space* $\mathbb{P}^n(\mathbb{K})$. If we take δ to be a more general *weighted degree*, we end up with the corresponding *weighted projective space*.

Both the usual and weighted degrees satisfy property 2 with exact *equality* instead of the inequality. We call the degree-like functions which have this property *semidegrees*. A classical example of a class of degree-like functions which are not semidegrees comes from *toric geometry* - where one associates a normal n-dimensional projective variety to a convex integral polytope of dimension n. Each facet (i.e. codimension one face) of such a polytope $\mathcal{P}$ defines a weighted degree on $\mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. It turns out [17, Example 3.4] that the toric variety $X_{\mathcal{P}}$ associated to $\mathcal{P}$ is precisely the completion of the n-torus $(\mathbb{K}^*)^n$ corresponding to the degree-like function $\delta_{\mathcal{P}}$ which is the maximum of the weighted degrees corresponding to the facets of $\mathcal{P}$. In our terminology, $\delta_{\mathcal{P}}$ is an example of a *subdegree* - a degree-like function which is the maximum of finitely many semidegrees.

Guiding principle: generalization of toric completions by means of subdegrees determined by semidegrees. One of the principles guiding me throughout this peoject is the conviction that the completion of $\mathbb{K}^n$ coming from a semidegree should 'behave similarly' to the weighted projective spaces, which are completions of $\mathbb{K}^n$ corresponding to weighted degrees. As the analogue of the $\delta_{\mathcal{P}}$ corresponding to a polytope $\mathcal{P}$, the same principle leads us to consider subdegrees. The first evidence for the validity of this principle comes from the following

Theorem 1.1 (The structure theorem for subdegrees [17, Theorem 4.1])**.** *The* minimal *presentation of a subdegree δ (on the coordinate ring $\mathbb{K}[X]$ of an arbitrary affine variety X) as the maximum of finitely many semidegrees is unique. Moreover, in the corresponding completion $\bar{X}^{\delta}$ of X, the irreducible components of the* hypersurface at infinity $X_\infty := \bar{X}^{\delta} \setminus X$ *have a canonical one-to-one correspondence with the non-trivial semidegrees associated to δ, in the same way that the irreducible components of $X_{\mathcal{P}} \setminus (\mathbb{K}^*)^n$ correspond to the facets of $\mathcal{P}$.*

It turns out that $\bar{X}^{\delta}$ is *relatively normal at infinity with respect to* X [17, Proposition 5.7], which means that for all open subset U of $\bar{X}^{\delta}$, the ring of regular functions on U is integrally closed in the ring of functions which are regular on $X \cap U$. A consequence of it is that $\bar{X}^{\delta}$ is *non-singular in codimension one at infinity* [17, Proposition 5.1], i.e. the codimension in $\bar{X}^{\delta}$ of $X_\infty \cap \operatorname{Sing}\bar{X}^{\delta}$ is at least two (where $\operatorname{Sing}\bar{X}^{\delta}$ is the set of singular points of $\bar{X}^{\delta}$). Given a completion $X \hookrightarrow \bar{X}^{\delta}$ determined by an arbitrary degree-like function δ, we introduce a 'normalization' procedure to produce a subdegree $\tilde{\delta}$ and a finite morphism $\phi : X^{\tilde{\delta}} \to \bar{X}^{\delta}$ such that ϕ is identity on X [17, Theorem 5.12]. Moreover, $X^{\tilde{\delta}}$ has the following universal property: if $\psi : Y \to \bar{X}^{\delta}$ is a dominant morphism of algebraic varieties such that Y is *relatively normal at infinity with respect to* X (via ψ), then there is a unique lift of ψ to $X^{\tilde{\delta}}$, i.e. there is a commuting diagram as follows:

$$\begin{array}{ccc} & & X^{\tilde{\delta}} \\ & \overset{\exists !}{\nearrow} & \downarrow \phi \\ Y & \xrightarrow{\psi} & \bar{X}^{\delta} \end{array}$$

We say that $X^{\tilde{\delta}}$ is the *normalization of $\bar{X}^{\delta}$ at infinity (with respect to X).* In particular, when X is normal, then $X^{\tilde{\delta}}$ is the normalization of $\bar{X}^{\delta}$. The construction of $\tilde{\delta}$ from δ generalizes the well known procedure of constructing the normalization of a non-normal toric variety (determined by a finite subset of a lattice) by 'filling the holes' [2, Theorem 3.A.5].

Finite generation of degree-like functions: Let A be a finitely generated algebra over $\mathbb{K}$. A degree-like function on A is called *finitely generated* if the corresponding graded ring is also a finitely generated algebra over $\mathbb{K}$. A basic building block of the theory of toric varieties is *Gordan's lemma* (see, e.g. [3, Proposition 1, Section 1.2]), which says that the semigroup of integral points in a convex rational cone in $\mathbb{R}^n$ is finitely generated. Another equivalent formulation of Gordan's lemma is that the maximum of finitely many weighted degrees (in $(x_1,\dots,x_n)$) on $\mathbb{K}[x_1,\dots,x_n]$ is finitely generated. The analogous question, which arises naturally in the context of the structure theorem of subdegrees (theorem 1.1) is the following:

Question 1. Is the maximum of finitely many finitely generated semidegrees also finitely generated?

In the case that $\dim A = 2$, we give (in [17, Theorem 6.9]) a positive answer to question 1 under the additional condition that the degree zero component of the graded ring corresponding to each semidegree is $\mathbb{K}$. It is a work in progress to better understand the necessity of the latter condition and generalize the result to higher dimensions.

Linking numbers at infinity: The negative of a semidegree is precisely a *discrete valuation*. More specifically, let δ be a finitely generated subdegree on $\mathbb{K}[X]$ with associated semidegrees $\delta_1,\dots,\delta_N$. For each i, $1 \le i \le N$, let d_{δ_i} be the positive generator of the subgroup of $\mathbb{Z}$ generated by $\{\delta_i(f) : f \in \mathbb{K}[X]\}$. Then I show in [17, Proposition 5.1] that for every i, $1 \le i \le N$, $\eta_i := -\frac{\delta_i}{d_{\delta_i}}$ is the order of vanishing along the component of the hypersurface at infinity of $\bar{X}^{\delta}$ associated to δ_i (where the association is given by the structure theorem of subdegrees). Moreover, d_{δ_i}'s are inversely proportional to the coefficients of irreducible Weil divisors in the expansion for the *divisor at infinity*. The divisor at infinity is an analogue of the *divisor of a polyhedron* in toric geometry (see, e.g. [2, Section 7.1]). I found a formula [17, Proposition 5.18] for the 'infinite part' of the pull-back of the divisor at infinity under a dominant morphism between completions of affine varieties corresponding to subdegrees. In connection with defining the pull-back of Weil divisors under birational regular mappings, P. Samuel introduced in [22] the notion of *linking numbers* of two discrete valuations. In [6] the definition of linking numbers was generalized to the case of *pseudo-valuations*. The coefficients of the irreducible Weil divisors in our pull-back formula turn out to be the inverses of the linking numbers (in the sense of [6]) of the (negative of the) corresponding degree-like functions. We refer to these coefficients as the *linking numbers at infinity* of corresponding degree-like functions.

Analogous to the matrix of intersection numbers of curves, in the set up of the preceding paragraph, we may form the *matrix L_δ of linking numbers at infinity*. More precisely, L_δ is the $N \times N$ matrix with entries $l_{ij} :=$ the linking number at infinity of δ_i and δ_j, $1 \le i,j \le N$. The diagonal entries of L_δ are all 1's and $l_{ij}l_{ji} > 1$ for all $i \neq j$, $1 \le i,j \le N$. A simple consequence of this observation is that if $N = 2$, then L_δ is invertible. This motivates us to ask the following question:

Question 2. Is L_δ invertible for all δ?

When $\dim X = 2$ and δ_j's are 'complete', i.e. they satisfy the hypothesis of our generalization of Gordan's lemma, I show that the answer to question 2 is indeed positive [17, Proposition 6.12], analogous to the invertibility of the matrix of intersection numbers of -1 *curves* on surfaces. A somewhat amusing (and easy to prove directly) consequence (which follows from taking $X = \mathbb{K}^2$ and δ_j's to be weighted degrees) is the following:

> Let $k \geq 1$ and $v_1, \ldots, v_k$ be mutually non-proportional elements of $\mathbb{Q}^2$ with positive coordinates. Let L be the $k \times k$ matrix with entries $l_{ij} := \max\{v_{ik}/v_{jk} : 1 \leq k \leq 2\}$. Then L is invertible. $(*)$

The preceding observation $(*)$ leads naturally to

Question 3. Does $(*)$ remain true with $\mathbb{Q}^2$ being replaced by $\mathbb{Q}^m$ for all $m \geq 1$?

In fact, a positive answer to question 3 is equivalent to a positive answer to question 2 for complete semidegrees and all dimensions. It is a work in progress to settle questions 2 and 3 in higher dimensions.

1.3 A framework for affine Bezout-type theorems

As stated in section 1.1, we denote by *affine Bezout-type* theorems the results which count number of solutions of systems of polynomials on an affine variety. All the affine Bezout-type theorems (see, e.g. [9], [1], [4], [5], [11], [20], [19], [21]) count the number of solutions of *generic* (in some suitable sense) systems of polynomial equations $f : X \to \mathbb{K}^n$ on the affine variety X under consideration. Moreover, underlying the proofs of most of these theorems there are constructions of suitable completions $X \hookrightarrow Z$ of X which satisfy the following property:

$$\text{for generic } a := (a_1, \ldots, a_n) \in \mathbb{K}^n,\ \bar{H}_1(a) \cap \cdots \cap \bar{H}_n(a) \cap (Z \setminus X) = \emptyset, \qquad (*)$$

where $H_i(a) := \{x \in X : f_i(x) = a_i\}$ and $\bar{H}_i(a)$ is the closure of $H_i(a)$ in Z for all $1 \leq i \leq n$. In other words, the completion Z of X does not add any 'parasite solution at infinity' to the system $f_1 - a_1, \ldots, f_n - a_n$ for generic $a := (a_1, \ldots, a_n) \in \mathbb{K}^n$. In my thesis, I showed that

Theorem 1.2 (Main Existence Theorem, [18])**.** *If $f : X \to \mathbb{K}^n$ is a generically finite map of affine varieties, then there is a subdegree δ on $\mathbb{K}[X]$ such that the corresponding completion $\bar{X}^\delta$ of X does not add any 'parasite solution at infinity' to the system $f_1 - a_1, \ldots, f_n - a_n$ for generic $a := (a_1, \ldots, a_n) \in \mathbb{K}^n$.*

In the set up of theorem 1.2, I prove an upper bound [18, Theorem 3.3.2] for the number of solutions of f in terms of the degree of $\bar{X}^\delta$ and show that this bound is exact when δ is a semidegree or $\dim X = 2$. It is an ongoing work to understand this estimate in higher dimensions.

Iterated semidegrees: In my thesis I studied a special class of semidegrees called *iterated semidegrees* which are inductively built from a starting semidegree. In the case that $X = \mathbb{K}^n$ and δ of theorem 1.2 is an iterated semidegree, I proved in the thesis an explicit and constructive formula for the degree of $\bar{X}^\delta$ and hence the number of solutions of f. Currently I am working on extending this formula to the case of subdegrees which are maxima of finitely many iterated semidegrees and which, I think, are sufficient for the purpose of theorem 1.2 for *any* f.

A constructive criterion for deciding on a suitable subdegree: I am trying to find a 'constructive criterion' of when a given completion of an affine variety X does not 'add any parasite solution at infinity' to a given polynomial system on X. Variants of this condition and its implications have been extensively studied in the case that X is the n-torus $(\mathbb{C}^*)^n$ and the completion is a toric variety corresponding to a polytope $\mathcal{Q}$ which is *compatible* with the Newton polytopes $\mathcal{P}_1,\dots,\mathcal{P}_n$ of the given (Laurent) polynomials (which means that the dual fan of $\mathcal{Q}$ subdivides the dual fan of $\mathcal{P} := \mathcal{P}_1 + \cdots + \mathcal{P}_n$), see e.g. [8], [19]. In particular, the above condition in this case is equivalent to the Kushnirenko-Bernstein non-degeneracy condition (see, e.g. [18, Theorem A.1]). To start with, I would like to have a similar understanding of the condition when X is the torus and $\mathcal{Q}$ is *not* necessarily compatible with $\mathcal{P}_i$'s. My hope is that it will lead to a better understanding of what the condition means in the case of a completion induced by a general subdegree in terms of the 'leading forms' of the polynomials corresponding to the associated *semidegrees*, generalizing the classical non-degeneracy condition of Kushnirenko-Bernstein. In the long run this will be valuable for finding an explicit algorithm to construct completions of affine varieties which do not add any parasite solution at infinity to a given system of polynomial equations (if the construction is explicit enough, as in the case of *iterated semidegrees*, it would lead to an explicit and exact count of the number of solutions of the given system, cf. [18, Theorem 3.2.7]).

Convex bodies associated to subdegrees: Recent works by K. Kaveh and A. Khovanskii [7], R. Lazarsfeld and M. Mustata [10] and others explore a beautiful connection between intersection theory of algebraic varieties and geometry of convex bodies. I have used the construction of [7] to give an interpretation of the degree D (in an appropriate projective space) of the completion of an affine variety by a degree like function in terms of the volume of an associated convex body [18, Proposition 3.1.5]. This number D also appears as a factor of the Bezout-type formula [18, Theorem 3.1.1] for semidegrees. In [18, Theorem 3.3.2] I generalize the Bezout-type formula for semidegrees to the case of more general subdegrees and there is a similar term D' in the latter formula as well. Let δ be a subdegree on the coordinate ring of an affince variety X with associated semidegrees $\{\delta_i : 1 \leq i \leq N\}$. I show in [13] that if $\dim X = 2$, then under a natural condition (that the valuation used to construct the convex body corresponding to δ 'maximally separates' the convex bodies corresponding to δ_i's), D' is in fact the *mixed coefficient* of a degree-2 homogeneous polynomial H defned in terms of 2 planar convex bodies $\{\Delta_1,\Delta_2\}$ constructed from δ_i's. If Δ_1 and Δ_2 are polygons, then $H(t_1,t_2)$ agrees with the area of $t_1\Delta_1 + t_2\Delta_2$ for all $t_1,t_2 \geq 0$ and as a result D' equals the *mixed volume* of Δ_1 and Δ_2. In particular, this implies Bernstein's theorem in dimension 2. But, as it is shown in [10], a priori Δ_1 and Δ_2 may not be polygons. On the other hand, the fact that H is a polynomial imposes severe restrictions on the shapes of the convex bodies associated to δ_i's. It is a current project of mine to better understand this situation and possibly generalize the formula to higher dimensions.

1.4 Completions of affine surfaces

The valuation theoretic aspects of the construction of *iterated semidegrees* (introduced in section 1.3) on polynomial rings are well studied (it is known as construction of valuations by 'key polynomials' and was introduced in [12]), but there is no general criterion for the finite generation of the graded ring associated to an iterated semidegree δ on a polynomial ring A (recall that the graded ring is $A^\delta := \bigoplus_{d \geq 0}\{f : \delta(f) \leq d\}$). In a forthcoming article [14] I show that in the case that $\dim A = 2$, every semidegree on A that corresponds to a component of the hypersurface at infinity on a completion of $\mathbb{K}^2$ is a *finitely generated* iterated semidegree (i.e. its associated graded ring is finitely generated). Currently I am trying to understand the implicaitons of this result on completions of $\mathbb{K}^2$ and how much of these hold for general affine surfaces.

2 Future Plan

Apart from the ongoing works described above, I intend to pursue the following projects in near future. (These, however, will not prevent me from working on new interesting problems!)

1. Let δ be a degree-like function on the coordinate ring $A := \mathbb{K}[X]$ of an affine variety X. Recall that the corresponding completion of X is $\bar{X}^\delta := \operatorname{Proj} \bigoplus_{d \geq 0} A^\delta_d$, where $A^\delta_d := \{f \in A : \delta(f) \leq d\}$. It turns out that the complement $\bar{X}^\delta_\infty$ of X in $\bar{X}^\delta$ is isomorphic to $\operatorname{Proj} \operatorname{gr} A^\delta$ (see, e.g. [18, Theorem 2.2]). Consider the isomorphism $A^\delta \cong \sum_{i \geq 0} A^\delta_i t^i$, where t is an indeterminate over A. Then $X \cong V(t-1)$ and $\bar{X}^\delta_\infty = V(t)$ as subvarieties of $\operatorname{Spec} A^\delta$, and as t goes from 1 to 0, we obtain a deformation of X to $\bar{X}^\delta_\infty$. In tropical geometry one considers limits (in the logarithmic scale) of algebraic varieties under a deformation associated to a toric completion. Whenever the limit exists and a 'topology' of the deformed variety remains unchanged for sufficiently small values of the parameter, one can find 'topological information' about the original variety by studying the (simpler) 'topology' of the limiting one. Since the topology of $\bar{X}^\delta_\lambda := V(t-\lambda)$ also remains unchanged for $0 < \lambda \leq 1$, this gives a motivation for finding δ so that $\bar{X}^\delta_\infty$ is a 'good topological limit'. To that end I plan to work on a project to develop a 'brand' of tropical geometry closely associated with my theory of projective completions determined by subdegrees.

2. The work of my thesis is a natural extension of the classical approach to tackle the problem of counting the number of solutions of a given polynomial system on an affine variety X by means of intersection theory on suitable compactifications of X (the difference between my approach and the classical approach is that classically one starts with a given completion and then tries to solve the problem for *generic* systems of polynomials for which this completion adds no 'parasite solutions at infinity', whereas I start with a given polynomial system with finitely many solutions and then try to find a compactification which adds no parasite solutions at infinity to this system). On the other hand, the theory of Gröbner basis provides an explicit algorithm to find the number of solutions (at least when all solutions of the given system are isolated). The connection between these two classical approaches are to the best of my knowledge hitherto completely unexplored. I believe that Buchberger's algorithm to find a Gröbner basis should have a geometric interpretation in terms of intersection theory and this I plan to explore.

References

[1] D. N. Bernstein. The number of roots of a system of equations. *Funkcional. Anal. i Priložen.*, 9(3):1–4, 1975.

[2] David Cox, John Little, and Hal Schenck. *Toric Varieties*. In preparation. Preliminary version (as of October 2009): http://www.cs.amherst.edu/∼dac/toric.html.

[3] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[4] Birkett Huber and Bernd Sturmfels. A polyhedral method for solving sparse polynomial systems. *Math. Comp.*, 64(212):1541–1555, 1995.

[5] Birkett Huber and Bernd Sturmfels. Bernstein's theorem in affine space. *Discrete Comput. Geom.*, 17(2):137–141, 1997.

[6] James A. Huckaba. Some results on pseudo valuations. *Duke Math. J.*, 37:1–9, 1970.

[7] Kiumars Kaveh and A. G. Khovanskii. Convex bodies and algebraic equations on affine varieties. arXiv:[math.AG], 2008.

[8] A. G. Khovanskii. Newton polyhedra, and toroidal varieties. *Funkcional. Anal. i Priložen.*, 11(4):56–64, 96, 1977.

[9] A. G. Kushnirenko. Newton polyhedra and Bezout's theorem. *Funkcional. Anal. i Priložen.*, 10(3, 82–83.), 1976.

[10] R. Lazarsfeld and M. Mustata. Convex bodies associated to linear series. arXiv:[math.AG], 2008.

[11] T. Y. Li and Xiaoshen Wang. The BKK root count in $\mathbf{C}^n$. *Math. Comp.*, 65(216):1477–1484, 1996.

[12] Saunders MacLane. A construction for absolute values in polynomial rings. *Trans. Amer. Math. Soc.*, 40(3):363–395, 1936.

[13] Pinaki Mondal. General bezout-type theorems on affine varieties. In preparation.

[14] Pinaki Mondal. On normal projective completions of the affine plane. In preparation.

[15] Pinaki Mondal. An affine Bezout type theorem and projective completions of affine varieties. *C. R. Math. Acad. Sci. Soc. R. Can.*, 30(4):1–12, 2008.

[16] Pinaki Mondal. On projective completions of affine varieties determined by 'degree-like' functions. arXiv:0901.4823v1 [math.AG], 2009.

[17] Pinaki Mondal. Projective completions of affine varieties determined via 'degree-like' functions. submitted in the Journal of Algebraric Geometry, 2010.

[18] Pinaki Mondal. *Towards a Bezout-type theory of affine varieties*. PhD thesis, University of Toronto, 2010.

[19] J. M. Rojas. Toric intersection theory for affine root counting. *J. Pure Appl. Algebra*, 136(1):67–100, 1999.

[20] J. Maurice Rojas. A convex geometric approach to counting the roots of a polynomial system. *Theoret. Comput. Sci.*, 133(1):105–140, 1994. Selected papers of the Workshop on Continuous Algorithms and Complexity (Barcelona, 1993).

[21] J. Maurice Rojas and Xiaoshen Wang. Counting affine roots of polynomial systems via pointed Newton polytopes. *J. Complexity*, 12(2):116–133, 1996.

[22] Pierre Samuel. Multiplicités de certaines composantes singulières. *Illinois J. Math.*, 3:319–327, 1959.