An integrable connection on the configuration space of a Riemann surface of positive genus

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ABSTRACT. Let $X$ be a Riemann surface of positive genus. Denote by $X^{(n)}$ the configuration space of $n$ distinct points on $X$. We use the Betti-de Rham comparison isomorphism on $H^1(X^{(n)})$ to define an integrable connection on the trivial vector bundle on $X^{(n)}$ with fiber the universal algebra of the Lie algebra associated to the descending central series of $\pi_1$ of $X^{(n)}$. The construction is inspired by the Knizhnik-Zamolodchikov system in genus zero and its integrability follows from Riemann period relations.

Fix $n \geq 1$. Let $g_0$ be the graded complex Lie algebra associated to the descending central series of the classical pure braid group $\text{PB}_n$, i.e. the fundamental group of

$$\mathbb{C}^{(n)} := \{(z_1, \ldots, z_n) : z_i \in \mathbb{C}, z_i \neq z_j \text{ for } i \neq j\}.$$ 

It is generated by degree 1 elements $\{s_{ij} : 1 \leq i,j \leq n, i \neq j\}$, subject to the relations

$$s_{ij} = s_{ji}$$

$$[s_{ij}, s_{kl}] = 0 \quad (i,j,k,l \text{ distinct})$$

$$[s_{ij} + s_{ik}, s_{jk}] = 0.$$ 

(1)

The element $s_{ij} \in H_1(\mathbb{C}^{(n)}, \mathbb{C})$ (= degree 1 part of $g_0$) is the homology class of the $j$-th strand going positively around the $i$-th, while all other strands stay constant.

Let $\wedge U_{g_0}$ be the completion of the universal algebra of $g_0$. Let $\mathcal{O}(\mathbb{C}^{(n)})$ (resp. $\Omega^1(\mathbb{C}^{(n)})$) be the space of analytic functions (resp. complex of holomorphic differentials) on $\mathbb{C}^{(n)}$. The relations (1) assure that the Knizhnik-Zamolodchikov connection

$$\nabla_{\text{KZ}} : \wedge U_{g_0} \otimes \mathcal{O}(\mathbb{C}^{(n)}) \to \wedge U_{g_0} \otimes \Omega^1(\mathbb{C}^{(n)})$$

defined by

$$\nabla_{\text{KZ}} f = df - \sum_{i<j} \frac{1}{2\pi i} s_{ij} \otimes \frac{d(z_i - z_j)}{z_i - z_j} f$$

is integrable. This connection and its more general variants are of great importance in conformal field theory, representation theory, and number theory.
The connection $\nabla_{KZ}$ is related to the comparison isomorphism
\[
\text{comp}_{\mathbb{C}^{(n)}} : H^1(\mathbb{C}^{(n)}, \mathbb{C}) \to H^1_{\text{dr}}(\mathbb{C}^{(n)})
\]
between the singular and (say) complex-valued smooth de Rham cohomologies in the following way: $\lambda_0$ is the image of $\text{comp}_{\mathbb{C}^{(n)}}$ under the map
\[
H_1(\mathbb{C}^{(n)}, \mathbb{C}) \otimes H^1_{\text{dr}}(\mathbb{C}^{(n)}) \to \bigwedge^2 U_{\mathbb{C}^{(n)}} \otimes \Omega^1(\mathbb{C}^{(n)})
\]
defined by
\[
s_{ij} \otimes \frac{d(z_k - z_l)}{z_k - z_l} \mapsto s_{ij} \otimes \frac{d(z_k - z_l)}{z_k - z_l} \quad (i < j, k < l).
\]
(Note that the $s_{ij}$ with $i < j$ (resp. $\frac{d(z_k - z_l)}{z_k - z_l}$ with $k < l$) form a basis of $H_1(\mathbb{C}^{(n)}, \mathbb{C})$ (resp. $H^1_{\text{dr}}(\mathbb{C}^{(n)})$).

Now let $\overline{X}$ be a compact Riemann surface of genus $g > 0$, $S = \{Q_1, \ldots, Q_{|S|}\}$ a finite set of points in $\overline{X}$ (possibly empty), and $X = \overline{X} - S$. Let
\[
X^{(n)} := \{(x_1, \ldots, x_n) : x_i \in X, x_i \neq x_j \text{ for } i \neq j\}.
\]
Fix a base point $e = (e_1, \ldots, e_n) \in X^{(n)}$ and let $g$ be the graded complex Lie algebra associated to the descending central series of $\pi_1(X^{(n)}, e)$. The goal of this note is to use the comparison isomorphism
\[
\text{comp}_{X^{(n)}} : H^1(X^{(n)}, \mathbb{C}) \to H^1_{\text{dr}}(X^{(n)})
\]
to define an integrable connection $\nabla$ on the trivial bundle $U_{\mathbb{C}^{(n)}} \otimes O(X^{(n)})$.

### 1. Construction of the connection

We make three observations first:

(i) Since $g > 0$, the natural map
\[
H^1_{\text{dr}}(X^n) \to H^1_{\text{dr}}(X^{(n)})
\]
(induced by inclusion) is an isomorphism. Indeed, thanks to a theorem of Totaro [7, Theorem 1] one knows that the five-term exact sequence for the Leray spectral sequence for the constant sheaf $\mathbb{Z}$ and the inclusion $X^{(n)} \to X^n$ reads
\[
0 \to H^1(X^n, \mathbb{Z}) \xrightarrow{(3)} H^1(X^{(n)}, \mathbb{Z}) \to \mathbb{Z}/\langle [a,b] : 1 \leq a < b \leq n \rangle \xrightarrow{(\ast)} H^2(X^n, \mathbb{Z}) \to H^2(X^{(n)}, \mathbb{Z}),
\]
where the map $(\ast)$ sends 1 in the copy of $\mathbb{Z}$ corresponding to $(a, b)$ ($a < b$) to the class of the pullback of the diagonal $\Delta \subset X^2$ under the projection $p_{ab} : X^n \to X^2$ (defined in the obvious way). Since $g > 0$, the class of $\Delta$ has a nonzero $H^1(X) \otimes H^1(X)$ Künneth component (if $X = \overline{X}$ this is well-known and the noncompact case follows from the compact case in view of the functoriality of the class of the diagonal with respect to the inclusion $i : X^2 \to X^n$ and injectivity of $i^* : H^2(X^n) \to H^2(X^2)$ on $H^2 \otimes H^1$ components). Thus the class of $p_{ab}^\ast(\Delta)$ has a nonzero $p_{ab}^\ast(H^1(X) \otimes H^1(X))$ component. Since every other $p_{ab}^\ast(\Delta)$ has a zero $p_{ab}^\ast(\Delta)$ component, it follows that $(\ast)$ is injective.

(ii) Let $\Omega^1(\overline{X} \log S)$ be the space of differentials of the third kind on $\overline{X}$ with singularities in $S$. Then one has a distinguished isomorphism $\Omega^1(\overline{X} \log S) \cong F^1H^1_{\text{dr}}(X)$ given by $\omega \mapsto [\omega]$ ($F^1$ being the Hodge filtration). (See [5, (3.2.13)(ii) and (3.2.14)], for instance.)
(iii) The cohomology $H^1_{\text{dR}}(X)$ decomposes as an internal direct sum $F^1H^1_{\text{dR}}(X) \oplus H^{0,1}$ (where $H^{0,1} \subset H^1_{\text{dR}}(\overline{X}) \subset H^1_{\text{dR}}(X)$). Indeed, this is simply the Hodge decomposition in $X = \overline{X}$ case. As for the noncompact case, strictness of morphisms of mixed Hodge structures with respect to the Hodge filtration implies the two subspaces have zero intersection, and by (ii) and the Riemann-Roch theorem $\overline{F^1H^1_{\text{dR}}(X)}$ has dimension $g + |S| - 1$. The conclusion follows by a dimension count.

Let $\theta$ be the composition

$$H^1_{\text{dR}}(X^{(n)}) \cong H^1_{\text{dR}}(X^n) \cong \cdots \cong H^1_{\text{dR}}(X) \cong F^1H^1_{\text{dR}}(X) \cong \Omega^1(\mathcal{X} \log S) \cong \Omega^1(X^{(n)}),$$

where $(\cdot)$ is the sum of $n$ copies of the natural projection, and $(\cdot)$ is the sum of the pullbacks along projections $X^{(n)} \to X$. Note that the image of $\theta$ is contained in the subspace of closed forms, as it is contained in the subspace spanned by the pullbacks of holomorphic 1-forms on $X$ along the aforementioned projections. Let $\iota$ be the composition of the inclusion $H_1(X^{(n)}, \mathbb{C}) \subset g$ and the natural map $g \to \mathcal{U}g$. Denote by $\lambda$ the image of the comparison isomorphism (2) under the map

$$\iota \otimes \theta : H_1(X^{(n)}, \mathbb{C}) \otimes H^1_{\text{dR}}(X^{(n)}) \to \mathcal{U}g \otimes \Omega^1(X^{(n)}).$$

Define the connection

$$\nabla : \mathcal{U}g \otimes \mathcal{O}(X^{(n)}) \to \mathcal{U}g \otimes \Omega^1(X^{(n)})$$

by

$$\nabla(f) = df - \lambda f.$$

(Note that $\lambda$ multiplies with an element of $\mathcal{U}g \otimes \mathcal{O}(X^{(n)})$ through the multiplication in the universal algebra in the first factor and the algebra of differential forms in the second.)

2. Integrability

We prove that the connection $\nabla$ is integrable. Since $\lambda \in \mathcal{U}g \otimes \Omega^1_{\text{closed}}(X^{(n)})$, it is enough to show that

$$\lambda^2 \in \mathcal{U}g \otimes \Omega^2(X^{(n)})$$

is zero. For simplicity denote $d = \dim H_1(X, \mathbb{Z})$ (thus $d = 2g$ if $X = \overline{X}$ and $d = 2g + |S| - 1$ otherwise). Let $\{\alpha_i\}_{1 \leq i \leq d}$ be a basis of $H_1(X, \mathbb{Z})$ such that for $i \leq g$, $\alpha_i$ and $\alpha_{i+g}$ are (classes of) transversal loops around the $i$-th handle with $\alpha_i \cdot \alpha_{i+g} = 1$ in $H_1(\overline{X}, \mathbb{Z})$, and for $1 \leq i \leq |S| - 1$, $\alpha_{2g+i}$ is a simple loop going positively around the puncture $Q_i$, contractible in $X \cup \{Q_i\}$. Let $\{\omega_i\}_{1 \leq i \leq d}$ be 1-forms such that $\{\omega_i\}_{1 \leq i \leq g}$ form a basis for holomorphic differentials on $\overline{X}$, $\omega_{g+i} = \overline{\omega_i}$ for $i \leq g$, and $\omega_{2g+i}$ ($1 \leq i \leq |S| - 1$) is a differential of the third kind with residual divisor $\frac{1}{2\pi i}(Q_i - Q_{i+1})$. With abuse of notation we denote a differential form (resp. a loop) and its cohomology (resp. homology) class by the same symbol. Write the comparison isomorphism $\text{comp}_X \in H_1(X, \mathbb{C}) \otimes H^1_{\text{dR}}(X)$ as

$$\sum_{i,j} \pi_{ij} \alpha_i \otimes \omega_j.$$ (Here and in all the sums in the sequel, unless otherwise indicated the indices run over all their possible values.) The matrix $(\pi_{ij})_{ij}$ (with $ij$-entry $\pi_{ij}$) is the inverse of the matrix whose $ij$-entry is $\int_{\omega_i} \omega_j$, and is of the form

$$\begin{pmatrix} p^{-1} & 0 \\ I_{|S|-1} \end{pmatrix},$$
where $P$ is the matrix of periods of $X$ with respect to the $\omega_i$ and $\alpha_j$, and $I$ denotes the identity matrix.

Let $\{\alpha_i^{(k)}\}_{1 \leq k \leq n}$ be pure braids in $X$ with $n$ strands based at $g$ (= loops in $X^{(n)}$ based at $g$) such that the following hold:

(i) The only nonconstant strand in $\alpha_i^{(k)}$ is the one based at $e_k$.
(ii) For $i \leq g$, the strands of $\alpha_i^{(k)}$ and $\alpha_{i+g}^{(k)}$ based at $e_k$ are transversal loops around the $i$-th handle.
(iii) For $1 \leq i \leq |S| - 1$, the strand of $\alpha_{2g+i}^{(k)}$ based at $e_k$ is a simple loop going around $Q_i$.
(iv) The $k$-th projection $X^{(n)} \to X$ sends $\alpha_i^{(k)}$ to $\alpha_i$ in homology.

Let $\omega_i^{(k)}$ be the pullback of $\omega_i$ under the $k$-th projection $X^{(n)} \to X$. Then $\{\alpha_i^{(k)}\}$ and $\{\omega_i^{(k)}\}$ are bases of $H_1(X^{(n)}, \mathbb{C})$ and $H_{dR}^1(X^{(n)})$, and

$$\text{comp}_{X^{(n)}} = \sum_{i,j,k} \pi_{ij} \alpha_i^{(k)} \otimes \omega_j^{(k)}.$$

Let $F = \{1, \ldots, d\} - \{g + 1, \ldots, 2g\}$. Then

$$\lambda = \sum_{i \in F} \sum_{j,k} \pi_{ij} \alpha_i^{(k)} \otimes \omega_j^{(k)}.$$

We have

$$\lambda^2 = \sum_{i,j,k,k'} \pi_{ij} \pi_{i'j'} [\alpha_i^{(k)}, \alpha_i^{(k')}] \otimes \omega_j^{(k)} \wedge \omega_j^{(k')}.
= \sum_{i,j,k,k'} \pi_{ij} \pi_{i'j'} \alpha_i^{(k)} \alpha_i^{(k')} \otimes \omega_j^{(k)} \wedge \omega_j^{(k')}.$$

Simple calculations using Bellingeri’s description of $\pi_1(X^{(n)})$ given in [1, Theorems 5.1 and 5.2] (also see [2] for a misprint corrected) show that in $g$, for arbitrary distinct $k, k'$, $[\alpha_i^{(k)}, \alpha_i^{(k')}] = 0$ unless $i, i' \leq 2g$ and $|i - i'| = g$ (i.e. unless $\alpha_i^{(k)}$, $\alpha_i^{(k')}$ correspond to transversal loops going around the same handle), and moreover that

$$[\alpha_i^{(k)}, \alpha_{i+g}^{(k')}](i \leq g)$$

only depends on the set $\{k, k'\}$. (Note that one can take $\alpha_i^{(k)} \in \pi_1(X^{(n)})$ to be Bellingeri’s $A_{2l-1-d+k}$, $A_{2(l-g),d+k}$, or $A_{i,d+k}$ depending on whether $i \leq g$, $g < i \leq 2g$, or $2g < i \leq d$ respectively.)

Denoting (4) by $s_{kk'} (= s_{k'k})$, we thus have

$$\lambda^2 = \sum_{j,k,k'} \left( \sum_{i \leq g} \pi_{ij} \pi_{i+g,j'} - \pi_{i+g,j} \pi_{ij} \right) s_{kk'} \otimes \omega_j^{(k)} \wedge \omega_j^{(k')}$$

which is zero by Riemann period relations.

REMARKS. (1) In the case $X = X$, one can replace $g$ by the Lie algebra $\mathfrak{l}$ of the nilpotent completion of $\pi_1(X^{(n)})$. Thanks to a theorem of Bezrukavnikov [3] one knows similar relations to the ones in $g$ used above to prove integrability also hold in $\mathfrak{l}$.

(2) It would be interesting to relate the connection defined here with the one defined by Enriquez in [6] on configuration spaces of compact Riemann surfaces.
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References