MAT301 Groups and Symmetry

Some more practice problems

Oct 8, 2018

1. Below the Cayley table of a group G of order 4 with elements w, x, y, z is given, with a few entries missing. Answer the following questions (and provide justification).

	w	x	y	z
w		y	x	
x		z	w	
y		w		y
z				

(a) What is the identity element of G?
(b) Complete the table.
(c) Is G abelian?

2. Give an example of an infinite group in which every element has finite order.

3. Find the centre of $SL_2(\mathbb{R})$.

4. Find the centralizer in $GL_2(\mathbb{R})$ of the subgroup

$$D := \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R} - \{0\}\}.$$

(See question #12 of the practice problems appended to Assignment 2 for the definition of centralizer.)

5. (a) Let G be a group. Define a relation $\sim$ on G as follows: given $g, g' \in G$, we set $g \sim g'$ if and only if $g' = hgh^{-1}$ for some $h \in G$. (The element hgh^{-1} is called the conjugate of g by h.) Show that $\sim$ is an equivalence relation.

(b) What is $[e]$? More generally, what is $[g]$ if $g \in Z(G)$?

(c) Let $G = D_3$ and calculate the equivalence classes of the relation $\sim$.

6. (a) Show that $\mathbb{Z}/n$ is generated by $[a]$ (i.e. $\mathbb{Z}/n = \langle [a] \rangle$) if and only if $\gcd(a, n) = 1$.

(b) Find the elements of $\mathbb{Z}/18$ which have order 6.

7. Suppose G is a group of order 20. Let $g \in G$ be an element that satisfies $g^9 = e$. Show that $g = e$.

8. Let H be a subgroup of D_n that contains rotation by $2\pi/n$ and at least one reflection. Show that $H = D_n$.

9. Let G be a group of prime order p. Show that there is an element $g \in G$ such that $G = \langle g \rangle$.

10. Let G be a non-abelian group of order $2p$, where p is a prime number.

(a) Show that G contains elements g, h with $|g| = 2$ and $|h| = p$.

(b) Show that
$$G = \{e, h, h^2, \ldots, h^{p-1}, g, gh, gh^2, \ldots, gh^{p-1}\}.$$

11. Suppose G is a group which has subgroups H and K with $|H| = 10$ and $|K| = 21$.

(a) Show that G has at least 210 elements.

(b) Show that $H \cap K = \{e\}$.

Some suggestions: (Please read these only after spending some time thinking about the questions.)

1. Note that we have been told that this is the Cayley table of a *group*. Note that *if we know* G *is a group* and $gh = g$ for some elements $g, h \in G$, then h has to be the identity (why?). Similarly, if G is a group with identity element e and $gk = e$, then k has to be the inverse of g (you may not need the latter fact for this problem).

2. It might be useful to have question 1(e) of the Assignment 2 in mind.

7. Remember the corollary to Lagrange's theorem: if G is a finite group and $g \in G$, then $|g| \bigm| |G|$.

8. Lagrange's theorem

9. Let g be an element of G other than the identity. Consider $\langle g \rangle$. What are the possibilities for $|\langle g \rangle|$?

10. (a) That G must have an element of order 2 will only use the hypothesis that $|G|$ is even. Write G as $S \cup T$, where $S = \{g \in G : g^2 = e\}$ and $S = \{g \in G : g^2 \neq e\} = G - S$. Argue that the number of elements of T is even (note that if $g \in T$, then $g^{-1} \in T$ as well, and $g^{-1} \neq g$ (why)).

    In order to prove that there must be an element of order p using what we have learned so far, you will have to use $|G| = 2p$ and the non-abelian hypothesis. Use Lagrange's theorem to limit the possibilities for the order for the elements of G. Can every nontrivial element have order 2? (One of the problems on the first assignment might be useful.)

    (b) Show that no two of the listed elements can be equal.

11. (b) Note that $H \cap K$ is a subgroup of both H and K.