MAT301 Groups and Symmetry

Assignment 5

Due Monday Nov 26 at 11:59 pm
(to be submitted on Crowdmark)

Please write your solutions neatly and clearly. Note that due to time limitations, only some of the questions will be graded.

1. Consider the following subset of S_4:

$$H := \{e,\ (12)(34),\ (13)(24),\ (14)(23)\}.$$

(a) Show that H is a normal subgroup of S_4. To prove normality you may use the following fact without proof: For every $\sigma, \delta \in S_n$, the permutations δ and $\sigma\delta\sigma^{-1}$ have the same cycle type.

(b) Show that the distinct (left or right because H is normal) cosets of H are H, $(123)H$, $(132)H$, $(12)H$, $(13)H$, and $(23)H$.

(c) Fill in the blanks with one of H, $(123)H$, $(132)H$, $(12)H$, $(13)H$, and $(23)H$. (The operations take place in the quotient group S_4/H.)

 (i) $(143)H \cdot (324)H = \ldots\ldots$

 (ii) $(1234)H \cdot (12)H = \ldots\ldots$

(d) Show that $S_4/H \simeq S_3$ by defining an isomorphism $S_3 \to S_4/H$.

2. Consider the subgroups $H = \langle [4] \rangle$ and $K = \langle [-4] \rangle$ of $U(15)$.

(a) Find $|U(15)/H|$ and $|U(15)/K|$.

(b) Find the order of the element $[2]K$ of $U(15)/K$.

(c) Is $U(15)/K$ cyclic?

(d) Find the order of every element of $U(15)/H$. Is $U(15)/H$ cyclic?

3. (a) Let G be an abelian group. Let H be the subset of G consisting of all the elements of finite order. By Problem 1(e) of Assignment 2, H is a subgroup of G (it is sometimes called the *torsion* subgroup of G). Show that the quotient group G/H has no nontrivial element of finite order (i.e. that the only element of finite order in G/H is the identity).

(b) How many elements of order 3 does the quotient group $\mathbb{C}^\times/\mu_4$ have? (Prove your claim.)

4. Let n be a positive integer and p be a prime number. Write $n = p^c m$, where $c \geq 0$ and m are integers and $p \nmid m$ (thus p^c is the highest power of p that divides n). Let G be an abelian group of order n. Let

$$H := \{g \in G : \text{there is } \ell \geq 0 \text{ such that } g^{p^\ell} = e\}.$$

In other words, H consists of all the elements of G whose order is a power of p. Show that $|H| = p^c$. (The subset H defined above is called the p-*part* of G.)

5. The goal of this question is to introduce the construction of *direct product* of two groups. Let G and H be any groups. Recall that the Cartesian product $G \times H$ is the set

$$G \times H := \{(g, h) : g \in G \text{ and } h \in H\}.$$

We can use the binary operations on G and H to define a binary operation on $G \times H$: define

$$(g, h) \cdot (g', h') := (gg', hh').$$

(In other words, we multiply elements of $G \times H$ "component-wise".)

(a) Calculate
$$((123),[3])\cdot((12),[4])$$
in $S_4 \times \mathbb{Z}/5$. (Remember the operation in $\mathbb{Z}/5$ is addition.)

(b) Back to the general G and H, show that $G \times H$ with the operation defined above is a group with identity (e_G, e_H). (This group is called the *direct product* of G and H.)

(c) Find the order of the element $(g,h) \in G \times H$ in terms of the orders of g and h.

(d) Show that $\mathbb{Z}/m \times \mathbb{Z}/n$ is cyclic if and only if $\gcd(m,n) = 1$.

6. (a) Let G be an abelian group. Let H and K be subgroups of G with $H \cap K = \{e\}$. Show that the map
$$\phi : H \times K \to G$$
defined by $\phi((h,k)) = hk$ is an injective homomorphism. (The condition $H \cap K = \{e\}$ should come into play for injectivity. To prove that ϕ is a homomorphism you will use the abelian hypothesis.)

(b) Let G be a finite abelian group of order mn, where $\gcd(m,n) = 1$. Let H and K be subgroups of G of orders respectively m and n. Show that $G \simeq H \times K$.

(c) Let G be an abelian group of order $p^a q^b$, where p and q are distinct prime numbers and $a, b \geq 0$ are integers. Show that there are abelian groups H and K of orders p^a and q^b such that $G \simeq H \times K$. (Suggestion: Take H and K to be the p-part and q-part of G.)

Practice Problems: The following problems are for your practice. They are not to be handed in for grading. I suggest to do questions marked with * first.

1.* Find the flaw(s) in the following argument, which claims to prove that $10 \mid 24$.

"Define the homomorphism $\phi : S_4 \to S_4$ by $\phi(\sigma) = \sigma^2$. Then the kernel of ϕ consists of the identity, permutations of cycle types 2,2 and 2,1,1. In S_4, there are 3 permutations of type 2,2, and there are 6 permutations of type 2,1,1. Thus $\ker(\phi)$ contains 10 elements. Since the kernel of a homomorphism is a subgroup, $\ker(\phi)$ is a subgroup of S_4. By Lagrange's theorem, $|\ker(\phi)| \mid |S_4|$, i.e. $10 \mid 24$."

2.* Let $H \le G$. In class, by considering the equivalence relation $\sim$ defined on G by $g \sim g'$ if $g'^{-1}g \in H$ we showed the following two statements:

(a) For any $g, g' \in G$, either $gH = g'H$ or $gH \cap g'H = \emptyset$.
(b) For any $g, g' \in G$, one has $gH = g'H$ if and only $g'^{-1}g \in H$.

Prove these statements directly, without using the relation.

3.* Let G be a group and $H \le G$.

(a) Show that H is normal if and only if $gH = Hg$ for every $g \in G$.
(b) Suppose $[G : H] = 2$. Show that H is normal in G. (In words, prove that every subgroup of index 2 is normal.)

4. Give an example of groups $K \le H \le G$ such that K is a normal subgroup of H and H is a normal subgroup of G, but K as a subgroup of G is not normal.

5. Let G and H be groups.

(a) Show that the map $\iota : G \to G \times H$ defined by $\iota(g) = (g, e_H)$ is an injective homomorphism. (This is called the *embedding* (or natural embedding) of G in $G \times H$. There is similarly a map $H \to G \times H$ defined by $h \mapsto (e_G, h)$, called the embedding of H in $G \times H$.)
(b) Show that the map $\pi : G \times H \to G$ defined by $\pi((g, h)) = g$ is a surjective homomorphism. (This map is called *projection* to the first coordinate. We similarly have a homomorphism "projection to the second coordinate".)

6.* Let G and H be finite cyclic groups with $\gcd(|G|, |H|) = 1$. Let $g \in G$ and $h \in H$. Show that the following two statements are equivalent.

(i) $G = \langle g \rangle$ and $H = \langle h \rangle$
(ii) $G \times H = \langle (g, h) \rangle$

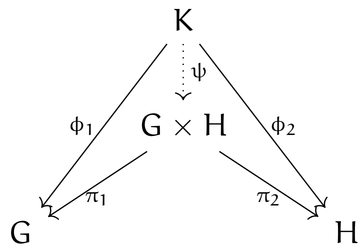
7. (a) Let $\gcd(m, n) = 1$. Show that $\varphi(mn) = \varphi(m)\varphi(n)$. (Here φ is Euler's function. Suggestion: Is $\mathbb{Z}/m \times \mathbb{Z}/n$ cyclic? Use the previous problem to count the number of generators of $\mathbb{Z}/m \times \mathbb{Z}/n$.)

(b) Let p be a prime number and $a \ge 1$. Show that $\varphi(p^a) = p^a - p^{a-1}$. (Don't try to use group theory here.)

(c) Find $\varphi(900)$. (Suggestion: First write 900 as a product of powers of distinct primes.)

8. Let G and H be groups, $K \le G$ and $L \le H$. Show that $K \times L$ is a subgroup of $G \times H$, and that $K \times L$ is normal in $G \times H$ if and only if K and L are respectively normal in G and H.

9.* (*universal property* of a direct product) (a) Let G and H be groups. Let $\pi_1 : G \times H \to G$ and $\pi_2 : G \times H \to H$ be the projection maps (defined respectively by $(g, h) \mapsto g$ and $(g, h) \mapsto h$). Let K be an arbitrary group. Let $\phi_1 : K \to G$ and $\phi_2 : K \to H$ be homomorphisms. Show that there exists a unique homomorphism $\psi : K \to G \times H$ such that $\pi_i \circ \psi = \phi_i$ for $i = 1, 2$. (Suggestion: Construct ψ. How about defining ψ to be "(ϕ_1, ϕ_2)"?)



(b) Can you reformulate what you proved above in terms of the three sets $\mathrm{Hom}(K, G)$, $\mathrm{Hom}(K, H)$, and $\mathrm{Hom}(K, G \times H)$?

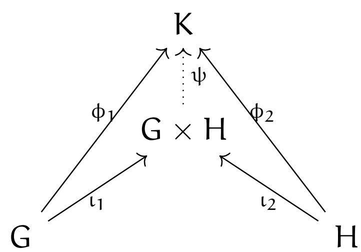
10.* (universal property of a direct "sum"†) Let G, H and K be abelian groups. Let $\phi_1 : G \to K$ and $\phi_2 : H \to K$ be homomorphisms.

(a) Show that the function

$$G \times H \to K$$

which sends $(g, h) \mapsto \phi_1(g)\phi_2(h)$ is a homomorphism. (Usually this map is denoted by $\phi_1\phi_2$, as you would expect.)

(b) Let ι_1 and ι_2 be the natural embeddings $G \to G \times H$ and $H \to G \times H$ (defined by $g \mapsto (g, e_H)$ and $h \mapsto (e_G, h)$ respectively). Show that there exists a unique homomorphism $\psi : G \times H \to K$ such that $\psi \circ \iota_1 = \phi_1$ and $\psi \circ \iota_2 = \phi_2$.



(c) Give a bijection

$$\mathrm{Hom}(G, K) \times \mathrm{Hom}(H, K) \to \mathrm{Hom}(G \times H, K).$$

11.* (a) true or false: Every quotient of a cyclic group is cyclic.
(b) true or false: Every quotient of an abelian group is abelian.
(c) true or false: The direct product $G \times H$ is abelian if and only if G and H are abelian.
(d) true or false: If the direct product $G \times H$ is cyclic, then G and H are cyclic.

12. Let G be a group with more than one element. Show that $\mathbb{Z} \times G$ is not cyclic.

13.* Find the Cayley table of the group $U(13)/H$, where $H = \langle 8 \rangle$. (First find the elements of $U(13)/H$ and then the table.)

14.* Show that the only element of $\mathbb{R}/\mathbb{Q}$ that has finite order is the identity element.

15. Show that the subgroup of $\mathbb{R}/\mathbb{Z}$ consisting of all the elements of finite order is $\mathbb{Q}/\mathbb{Z}$.

16.* Find all the elements of order 6 in $\mathbb{C}^\times/\mu_4$.

17. Let k and n be positive integers. How many elements of order k does $\mathbb{C}^\times/\mu_n$ have?

18.* Let H be a subgroup if index 2 in G. Let $g, g' \in G$. Show that if g, g' are both not in H, then $gg' \in H$. (Suggestion: Remember every subgroup of index 2 is normal. Work with the quotient G/H.)

19. Let $K \leq H \leq G$. Suppose $K \trianglelefteq G$.

(a) true or false: K is a normal subgroup of H.
(b) true or false: H/K is a subgroup of G/K.

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†If the groups G and H are abelian, their direct product is also referred to as their direct sum.

20.* (a) Let G be a group. Show that $Z(G) \trianglelefteq G$. (Recall that $Z(G)$ is the centre of G. By definition, $Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}$.)

(b) Suppose G is a group such that $G/Z(G)$ is cyclic. Show that G is abelian.

(c) Let G be a non-abelian group of order pq, where p and q are prime numbers. Show that the centre of G is the trivial subgroup.

21.* Let H be a normal subgroup of G of finite index. Let $g \in G$ be an element of finite order such that $\gcd(|g|, [G : H]) = 1$. Show that $g \in H$. (Suggestion: Consider the quotient map $G \to G/H$.)

22.* Let G be an abelian group of order pq, where p and q are distinct primes. Show that G is cyclic. Suggestion: Cauchy's theorem implies G contains an element g of order p and an element h of order q.)

23. Give an example of a group G that is not abelian, but $G/Z(G)$ is abelian. (Suggestion: Maybe D_4?)

24. true or false: For any group G, the quotient $G/[G, G]$ is abelian. (Here $[G, G]$ is the commutator subgroup of G, which is normal - see Problems 8 and 9 of the practice list appended to Assignment 4.)

25. (a) Suppose G is a divisible group. Show that G has no proper subgroup of finite index. (For the definition of what it means for a group to be divisible, see Problem 19 of the practice list in Assignment 4. Suggestion: Let $H \leq G$ be a proper subgroup of finite index. Is G/H a finite divisible group?)

(b) Conclude that every proper subgroup of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{C}^\times$ has infinite index.

(c) Give an example of an infinite abelian group which has a proper subgroup of finite index.

26. Let G be a group with n elements. Show that G is isomorphic to a subgroup of S_n. (Subgroups of the symmetric groups are called permutation groups. By what you prove in this exercise, every finite group is isomorphic to a permutation group. Note: This question is on the material before quotients. Suggestion: Let $\mathrm{Bij}(G)$ denote set of all bijective *functions* $G \to G$. Then $\mathrm{Bij}(G)$ is a group under composition. Do you agree that $\mathrm{Bij}(G) \simeq S_n$? Try to define an injective homomorphism $G \to \mathrm{Bij}(G)$. The construction is something you have seen before.)

27.* Suppose H is the unique subgroup of G of order n. Show that H is normal in G. (Suggestion: Let $g \in G$. Is $gHg^{-1} := \{ghg^{-1} : h \in H\}$ a subgroup of G of order n?)

28. Let $n \geq 2$. The goal of this question is to show that A_n is the only subgroup of index 2 of S_n. We shall do this in a few steps. Suppose $H \leq S_n$ and $[S_n : H] = 2$. Note that it follows automatically that $H \trianglelefteq S_n$. The quotient group S_n/H has order 2. Let us call its two elements e and g, where e is the identity. We have $g^2 = e$.

(a) Let $\delta, \delta' \in S_n$ be 2-cycles. Show that there is $\sigma \in S_n$ such that $\delta' = \sigma\delta\sigma^{-1}$. (One phrases this result by saying that every two 2-cycles are *conjugates* of one another.)

(b) Let $\pi : G \to G/H$ be the quotient map. Show that if $\ker(\pi)$ contains one 2-cycle, then it must contain every 2-cycle.

(c) Note that (b) implies either $\ker(\pi)$ contains every 2-cycle or it contains no 2-cycle. Argue that the former is impossible. (Hence $\pi(\delta) = g$ for every 2-cycle δ.)

(d) Show that H contains A_n.

(e) Conclude that $H = A_n$.

29.* In each case, determine if there is a homomorphism as described. If there is one, give an example. If there isn't, prove so. You may take the following theorem for granted: For $n > 1$ the group $U(n)$ is cyclic if and only if n is either 2, 4, a power of an odd prime number, or 2 times a power of an odd prime number.

(a) a surjective homomorphism $\mu_{16} \to U(15)$

(b) a surjective homomorphism $U(100) \to \mathbb{Z}/15\mathbb{Z}$
(c) a surjective homomorphism $\mathbb{C}^\times \to \mathbb{R}^\times$
(d) an injective homomorphism $U(15) \to \mathbb{Z}/40\mathbb{Z}$
(e) an injective homomorphism $D_5 \to S_4$
(f) a surjective homomorphism $U(15) \to \mathbb{Z}/4\mathbb{Z}$

30.* Let G be a group and $\phi : G \to H$ a surjective homomorphism with kernel K. Suppose there is a homomorphism $\rho : G \to K$ such that $\rho(k) = k$ for every $k \in K$. Show that $G \simeq K \times H$.

31.* Find the flaw(s) in the following argument which claims to prove that D_7 is abelian.

"Let s be a reflection in D_7 and $K = \langle s \rangle$. Then

$$|D_7/K| = [D_7 : K] = \frac{|D_7|}{|K|} = \frac{14}{2} = 7.$$

Being a group of prime order, D_7/K is cyclic.
Let r be a rotation of order 7 in D_7 and $L = \langle r \rangle$. Then

$$|D_7/L| = [D_7 : L] = \frac{|D_7|}{|L|} = \frac{14}{7} = 2.$$

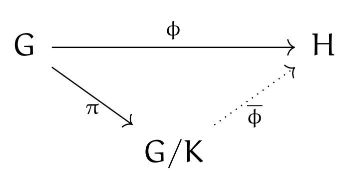
Note that $D_7/L = \{L, sL\}$. Being groups of order 2, there is a unique isomorphism $D_7/L \to K$ (namely, the map that sends identity to identity and sL to s). Let ρ be the composition

$$D_7 \xrightarrow{\text{quotient}} D_7/L \longrightarrow K,$$

where the second map is the isomorphism just described. Then being a composition of homomorphisms, ρ is a homomorphism. Moreover, we have $\rho(k)$ for every $k \in K$ (i.e. for $k = e, s$). It now follows from the previous exercise (applying to the quotient map $\pi : D_7 \to D_7/K$ and $\rho : D_7 \to \ker(\pi) = K$) that $D_7 \simeq K \times (D_7/K)$. The groups K and D_7/K are both cyclic and hence abelian, so that their direct product $K \times (D_7/K)$ is also abelian. Thus D_7 is abelian as well."

32. Let G be an abelian group and $\phi : G \to H$ a surjective homomorphism with kernel K. Suppose there is a homomorphism $\psi : H \to G$ such that $\phi \circ \psi : H \to H$ is the identity map on H. Show that $G \simeq K \times H$. (Suggestion: Try to define an isomorphism $K \times H \to G$. Where are you using the hypothesis that G is abelian?)

33. (universal property of a quotient) Let $K \trianglelefteq G$ and $\pi : G \to G/K$ be the quotient map. Let $\phi : G \to H$ be a homomorphism to an arbitrary group H. Suppose $K \subset \ker(\phi)$. Show that there exists a unique homomorphism $\overline{\phi} : G/K \to H$ such that $\phi = \overline{\phi} \circ \pi$. (In words, if $K \subset \ker(\phi)$, then ϕ "factors uniquely through G/K". Note that the condition that $K \subset \ker(\phi)$ is crucial. Suggestion: Uniqueness is easy once existence is done. For existence, construct $\overline{\phi}$. The condition that $K \subset \ker(\phi)$ is required to guarantee that $\overline{\phi}$ is well-defined.)



34. Let G be a group, K and H be normal subgroups of G, and $K \leq H$. Construct a natural surjective homomorphism $G/K \to G/H$ with kernel H/K.