1. Introduction

My research interests lie in the interplay between number theory and algebraic geometry. I am interested in the Hodge-de Rham realization of mixed motives (in particular, motivic fundamental groups), periods, algebraic cycles, and possible applications to rational points on varieties. The projects I have been working on until now can mostly be interpreted in the framework of Grothendieck’s period conjecture, the Hodge-Nori conjecture for mixed motives, and the conjectures of Beilinson and Bloch on algebraic cycles and cycle class maps.

This document is organized as follows. In the remainder of this introduction, I will briefly discuss my projects in rough terms, classified by theme. I then dedicate a section to recall the general philosophy of motives and the conjectures mentioned above; I hope that this will put my work in some context for a reader from another area of mathematics. I will then discuss two of my projects in more details in Sections 3 and 4.

1.1. Work on algebraic cycles and Hodge theory of fundamental groups.

1.1.1. In [Es19] (which contains the work mostly done in my PhD thesis), in the spirit of Pulte [Pu88], Kaenders [Ka01] and Darmon-Rotger-Sols [DRS12], and building on their ideas, I considered a certain extension of mixed Hodge structures that arises from the weight filtration on Hain’s mixed Hodge structure on $(\mathbb{Z}[\pi_1(U, e)]/\mathbb{Z}^{n+1})^\gamma$, where $U$ is a complex smooth projective curve $X$ minus a single point, $e \in U$ is a base point, $\pi_1$ is the classical fundamental group (for analytic topology), and $I$ is the augmentation ideal (the kernel of the map $\mathbb{Z}[\pi_1(U, e)] \rightarrow \mathbb{Z}$ sending $\gamma \mapsto 1$ for every $\gamma \in \pi_1(U, e)$). This extension can be naturally thought of as an element of $\text{Ext}((H^1)^{\otimes 2n-1}(n-1), \mathbb{Z}(0))$, where $\text{Ext}$ is the group of extensions in the category of mixed Hodge structures, and $H^1 = H^1(X) = H^1(U)$, and $M(n) = M \otimes \mathbb{Z}(n)$ as usual. The conjectures of Beilinson and Bloch predict that motivic extensions in $\text{Ext}((H^1)^{\otimes 2n-1}(n-1), \mathbb{Z}(0)) \subset \text{Ext}(H^{2n-1}(X^{2n-1})(n-1), \mathbb{Z}(0))$ should be (after tensoring with $\mathbb{Q}$) in the image of the Abel-Jacobi map for $\text{CH}_{n-1}^{\text{hom}}(X^{2n-1})$, where $\text{CH}$ is for the Chow group and the superscript hom is for the homologically trivial subgroup. The main result of [Es19] asserts that this is indeed the case for the particular extension above. The article also contains an application of this result to construction of rational points on the Jacobian of $X$ (assuming $X, X - U, e$ are all defined over a subfield $K$ of $\mathbb{C}$). See Section 3 for more details.

1.1.2. If the $K$-rational points on the Jacobian constructed in [Es19] are torsion, they give rise to relations between periods of iterated integrals on $U$. The de Rham structure on the complexification of $(\mathbb{Z}[\pi_1(U, e)]/\mathbb{Z}^{n+1})^\gamma = (\mathbb{C}[\pi_1(U, e)]/\mathbb{Z}^{n+1})^\gamma$, which by a theorem of K. T. Chen is the space of closed iterated integrals of length $\leq n$ on $U$) is given by iterated integrals formed by algebraic differential forms defined over $K$. To explicitly write the relations that one can get between the periods of the Hodge-de Rham structure $(\mathbb{Z}[\pi_1(U, e)]/\mathbb{Z}^{n+1})^\gamma$ one would like to have an explicit description of the Hodge filtration on $(\mathbb{C}[\pi_1(U, e)]/\mathbb{Z}^{n+1})^\gamma$ in terms of algebraic differential forms over $K$. In [Es18a] I carry out this approach for a punctured elliptic curve in the case $n = 2$ and as a result prove a relation between the quadratic and classical periods of algebraic 1-forms for such a curve. See the referenced article for more details.

1.1.3. In a more recent work [EM19a] joint with K. Murty, we combined the works [Pu88], [Ka01], and [DRS12] of Pulte, Kaenders, and Darmon-Rotger-Sols on Hodge theory of quadratic iterated integrals on algebraic curves with Gross-Röhrlich’s [GR78] on points of infinite order on Jacobians of Fermat curves to show that the Ceresa cycle of the Fermat curve of degree $n$ has Abel-Jacobi image of infinite order.
(as predicted by the conjectural injectivity of Abel-Jacobi maps mod torsion for varieties over number fields), if \( n \) has a prime divisor greater than 7. See [EM19a] for more details.

### 1.2. Work on Mumford-Tate groups of mixed Hodge structures

In a joint work with K. Murty, we look for ways to calculate the unipotent radical of the Mumford-Tate group of a mixed Hodge structure. In [EM19b], we give a rather computable characterization of the unipotent radical \( U(M) \) of the Mumford-Tate group of a polarizable mixed Hodge structure \( M \) with only two weights. If \( M \) has weights \( a < b \), the sequence

\[
0 \rightarrow Gr^W_a M \rightarrow M \rightarrow Gr^W_b M \rightarrow 0
\]

gives an extension \( \mu \in \text{Ext}(\text{Gr}^W_a M \wedge \text{Gr}^W_b M, \mathbb{Q}(0)) \). Roughly speaking, our result says to know \( U(M) \) it is enough to know which restrictions of \( \mu \) split. We then go on and apply this to \( H^1 \) of an open curve and the mixed Hodge structure on the space of quadratic iterated integrals on a curve. See Section 4.1 for a summary of [EM19b].

With Murty we would like to generalize [EM19b] to objects with more than two weights. This is now done for objects with “separated weights”, by which we mean objects with the property that the numbers \( n - m \) with \( m < n \), \( \text{Gr}^W_m \) and \( \text{Gr}^W_n \) both nonzero, are all distinct. See Section 4.2 for more details.

We should mention that the characterization of \( U(M) \) given in [EM19b] and its generalization to the multiple weight case also apply to the unipotent radical of the motivic Galois group of a mixed motive over a field of characteristic zero (with Ext groups now considered in the category of mixed motives).

The case of objects with unseparated weights seems complicated at the moment but we hope to be able to make some progress in this direction in the future.

### 1.3. Work towards analogues of the pentagon relation for periods of iterated integrals on higher genus curves

The multiple zeta values, i.e. numbers of the form

\[
\zeta(k_1, \ldots, k_r) := \sum_{0 < n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},
\]

can be expressed as iterated integrals on \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \). Thanks to the work of Furusho ([Fu10] and [Fu11]) the most general class of known polynomial relations between multiple zeta values (which are conjecturally expected to be all polynomial relations between them) are given by the so-called pentagon relation, a relation satisfied by the Drinfeld associator which was first discovered by Drinfeld [Dr91] by studying the monodromy of the Knizhnik-Zamolodchikov (KZ) system. The KZ system is a differential equation on the configuration space of \( n \) ordered marked points on \( \mathbb{P}^1 \) with only two weights. If \( n - m \) with \( m < n \), \( \text{Gr}^W_m \) and \( \text{Gr}^W_n \) both nonzero, are all distinct. See Section 4.2 for more details.

More precisely, it is an integrable connection

\[
\nabla_{\text{KZ}} : \mathcal{U}_{\mathfrak{g}_0} \otimes \mathcal{O}(\mathbb{C}^{[n]}) \longrightarrow \mathcal{U}_{\mathfrak{g}_0} \otimes \Omega^1(\mathbb{C}^{[n]})
\]

where \( \mathfrak{g}_0 \) is the graded complex Lie algebra associated to the descending central series of the fundamental group of \( \mathbb{C}^{[n]} \), and \( \mathcal{U}_{\mathfrak{g}_0} \) is the completion of the universal algebra of \( \mathfrak{g}_0 \), and \( \mathcal{O} \) and \( \Omega^1 \) are respectively the space of holomorphic functions and 1-forms on \( \mathbb{C}^{[n]} \).

Drinfeld’s method was later successfully generalized by B. Enriquez to \( \mathbb{P}^1 \) minus 0, \( \infty \) and \( n \)-th roots of unity (see [En07]). As a first step towards finding analogues of the pentagon relation for iterated integrals on a curve \( X \) of positive genus, in [Es18b] I used the Betti-de Rham comparison isomorphism on \( H^1(X^{[n]}) \) (where \( X^{[n]} \) is the configuration space of \( n \) ordered marked points on \( X \)) to define an integrable connection on the trivial bundle \( \mathcal{U}_{\mathfrak{g}} \otimes \mathcal{O}(X^{[n]}) \), where \( \mathfrak{g} \) is the graded complex Lie algebra associated to the descending central series of the fundamental group of \( X^{[n]} \). The construction of this connection is inspired by the KZ system and its integrability follows from Riemann’s period relations (see [Es18b] for details). In a future project I would like to explore whether a variant of this connection can be used to prove analogues of the pentagon relation for some curves of positive genus. (Note that my goal here would not be to generalize all aspects of the KZ system, but rather just to get relations between iterated integrals on higher genus curves.)
In the remainder of this document, after an overview of the philosophy of motives I will discuss two of the projects mentioned above, namely those previewed in Sections 1.1.1 and 1.2, in more details.

2. Digression: Reminder on motives and related conjectures

The goal of this section is to provide readers from other areas of mathematics some context for my research. Readers with some familiarity with cohomology theories for algebraic varieties and the philosophy of motives may skip this section.

Let \( K \) be a subfield of \( \mathbb{C} \). There are different cohomology theories attached to a smooth variety \( X \) over \( K \).

- **Singular (Betti) cohomology**: Thinking of \( X \) as a manifold (that is, with classical topology), one has the singular cohomology space \( H^n(X, \mathbb{Q}) \). This cohomology comes with an extra structure: in the case that \( X \) is projective (so that the manifold is compact Kähler), classical Hodge theory shows that \( H^n(X, \mathbb{Q}) \) underlies a Hodge structure of weight \( n \). This means one has a decomposition

\[
H^n(X, \mathbb{Q}) \otimes \mathbb{C} = H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}
\]

with \( H^{p,q} \) complex conjugates of one another. The decomposition is equivalent to the data of a finite decreasing filtration (Hodge filtration) \( F \) on \( H^n(X, \mathbb{C}) \) with \( H^n(X, \mathbb{C}) = F_0 \oplus F_{n-p+1} \) for each \( p \). In the general case, by Deligne’s work [De71], the space \( H^n(X, \mathbb{Q}) \) underlies a so-called mixed Hodge structure. This is the data of two finite filtrations \( W \) (increasing, called the weight filtration) and \( F \) (decreasing, called the Hodge filtration) on \( H^n(X, \mathbb{Q}) \) and the complexification \( H^n(X, \mathbb{C}) \), respectively, such that for each \( m \), the filtration that \( F \) induces on the complexification of \( Gr^W_m := W_m/W_{m-1} \) makes \( Gr^W_m \) a Hodge structure of weight \( m \).

- **Algebraic de Rham cohomology**: Grothendieck’s algebraic de Rham cohomology theory gives a space \( H^n_{dR,alg}(X) \), which is a vector space over \( K \). For instance, if \( X \) is a subset of \( \mathbb{C}^n \) cut out by polynomials with coefficients in \( K \), then this is simply the space of closed algebraic \( n \)-forms on \( X \) (i.e. closed differential forms on \( X \) of the form \( \sum_i f dz_1 \wedge \ldots \wedge dz_{n} \), with the \( f \) polynomials in the coordinates \( z_j \) \((1 \leq j \leq N)\) and with coefficients in \( K \)) modded out by exact such forms.

  We should mention that the algebraic de Rham cohomology is a cohomology theory defined using the Zariski topology.

- **Étale cohomology**: For every prime number \( \ell \), the étale cohomology with coefficients in \( \mathbb{Q}_\ell \) (= the \( \ell \)-adic completion of \( \mathbb{Q} \)) is a \( \mathbb{Q}_\ell \)-vector space together with an action of the Galois group \( \text{Gal}(\overline{K}/K) \) (where \( \overline{K} \) is an algebraic closure of \( K \)).

These cohomology spaces, while defined differently and even rising from different topologies, have the same dimensions. In fact, after suitable extensions of coefficients, one has canonical isomorphisms (called comparison isomorphisms) between them. For instance, one has a comparison isomorphism which relates the singular and algebraic de Rham cohomologies; this isomorphism, which is the comparison isomorphism more relevant to my work, simply comes from integration. Indeed, one can prove that \( H^n_{dR,alg}(X) \) can be naturally thought of as a K-subspace of the complex smooth de Rham cohomology \( H^n_{dR,\infty}(X) \) (= closed complex-valued smooth 1-forms modulo exact forms, with \( X \) considered as a manifold). To be more concrete, in the case where \( X \) is a subset of \( \mathbb{C}^N \) cut out by polynomials, thinking of algebraic forms as smooth forms, we have a map \( H^n_{dR,alg}(X) \rightarrow H^n_{dR,\infty}(X) \), which turns out to be injective. In general, \( H^n_{dR,alg}(X) \) has the same \( K \)-dimension as the complex dimension of \( H^n_{dR,\infty}(X) \). The integration pairing

\[
(2) \quad H_n(X, \mathbb{C}) \otimes \mathbb{C} H^n_{dR,\infty}(X) \rightarrow \mathbb{C}, \quad \gamma \otimes \omega \mapsto \int_\gamma \omega
\]

\footnote{For simplicity, let us think of a variety over \( K \) as a subset of \( \mathbb{C}^N \) or the projective \( N \)-space \( P^n_\mathbb{C} \) defined by polynomial equations with coefficients in \( K \).}
gives rise to the comparison isomorphism
\[ H^n(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = H^n(X, \mathbb{C}) \cong H^n_{dR,\infty}(X) = H^n_{dR,\text{alg}}(X) \otimes_{\mathbb{K}} \mathbb{C}. \]

The compatibility between different cohomology theories lead Grothendieck to start thinking about a sort of unifying cohomology theory for varieties over \( K \) (a theory of so called motives over \( K \)) in the sixties. He introduced the idea in a letter to Serre in 1964 [Gr-Se, page 195]. In Grothendieck’s vision, every degree cohomology of a variety (without specifying a particular cohomology theory) had a motive. The cohomology groups obtained in different cohomology theories were then the various realizations of this motive.

The category of motives was to be constructed geometrically from the category of varieties and satisfy certain requirements (e.g. being abelian). Grothendieck himself constructed a candidate for motives for smooth projective varieties (the pure case) using algebraic cycles (= formal linear combinations of closed irreducible algebraic subsets of a variety), which would satisfy the desired properties assuming Grothendieck’s Standard Conjectures on algebraic cycles.

Today, after decades of research (including [De82], [De89], [An96] [Ja90], [Vo00]), one finally has two equivalent unconditional categories of mixed motives due to Nori [No00] and Ayoub [Ay14b]. (“Mixed” means the category has nontrivial extensions, or that the Hodge realization functor gives mixed Hodge structures. This is the theory that allows non-projective varieties as well.) There are still several very deep open conjectures regarding mixed motives. These include:

- Grothendieck’s period conjecture: The numbers in the image of the subspace \( H^n(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^n_{dR,\text{alg}}(X) \) under the pairing of Eq. (2) are called the periods of \( H^n(X) \). More generally, for any mixed motive one similarly has a notion of periods. Grothendieck’s period conjecture concerns the transcendence degree \( d(M) \) of the field generated by the periods of a mixed motive \( M \). Assume \( K = Q \). The period conjecture predicts \( d(M) \) to be equal to the dimension of the motivic Galois group of \( M \), an algebraic group over \( Q \) associated to \( M \) via Tannakian formalism (see Section 4 for more details on the definition of this group).

To see the depth of this conjecture, let us look at the simplest nontrivial example. Consider the punctured complex plane \( C - \{0\} \). We can identify \( C - \{0\} \) with \( \{(z, w) \in C^2 : zw = 1\} \) (via \( z \mapsto (z, \frac{1}{z}) \)), so that \( C - \{0\} \) can be thought of as a variety over \( \overline{Q} \). The singular homology \( H_1(\mathcal{C} - \{0\}, \mathbb{Q}) \) is generated by a small loop around 0. The algebraic de Rham cohomology is generated by \( \frac{dz}{z} \), and the space of periods of \( H^1(\mathcal{C} - \{0\}) \) is the \( \overline{Q} \)-span of \( \int_{|z|=1} \frac{dz}{z} = 2\pi i \). On the other hand, the motivic Galois group of any motive \( M \) can be naturally thought of as a subgroup of the general linear group \( \text{GL}_{M_{B}} \), where \( M_{B} \) is the singular realization of \( M \). In the case of \( M = H^1(\mathcal{C} - \{0\}) \), we get that the motivic Galois group is a subgroup of \( \text{GL}_Q \). One easily sees that, in fact, here the motivic Galois group is equal to \( \text{GL}_Q \). Thus the period conjecture for \( H^1(\mathcal{C} - \{0\}) \) amounts to the transcendence of \( \pi \).

Grothendieck’s period conjecture is in general widely open. For instance, it is not even known for \( H^1 \) of an elliptic curve without complex multiplication. (It is known to hold for \( H^1 \) of an elliptic curve with complex multiplication, thanks to Chudnovsky [Ch76] (also see [An96] and [Ar13]). We refer the reader to [Ay14a] for a nice introduction to the period conjecture. We should mention that it is easy to show that \( d(M) \) is always bounded from above by the dimension of the motivic Galois group.

- Hodge-Nori conjecture: This is a generalization of a version of the Hodge conjecture. Suppose \( K \) is algebraically closed. The Hodge-Nori conjecture asserts that any morphism between the mixed Hodge structures of two motives is motivic, i.e. comes from a morphism in the category of motives. A slightly stronger version of it predicts that the motivic Galois group of a motive coincides with its so called Mumford-Tate group, which is an algebraic group associated to the mixed Hodge structure of the motive. (This group only depends on the mixed Hodge structure;
a priori, there is no reason for it to know anything about other realizations such as de Rham or étale. See Section 4 for more details on the definition of the Mumford-Tate group.)

The strong form of the Hodge-Nori conjecture is now known for 1-motives (these include motives of curves) thanks to [An19].

- The conjectures of Beilinson and Bloch: These are a number of deep conjectures regarding existence of a natural filtration $F$ on the space of algebraic cycles on a projective variety $X$, related to extensions in the category of mixed motives and values of $L$-functions. In particular, when $K$ is a number field, the conjectures predict that one has an isomorphism

$$\text{CH}^j(X)_{0,Q} \cong \text{Ext}^1_{MHS(K)}(Q(0), H^{2j-1}(X)(j)),$$

where $\text{CH}^j(X)_{0,Q}$ is the subspace of the rational Chow group (algebraic cycles of codimension $j$ with coefficients in $Q$ modulo so called rational equivalence) consisting of the homologically trivial cycles (i.e. those whose (co)homology class is zero), $\text{Ext}^1_{MHS(K)}$ is the Yoneda group of extensions in the category of mixed motives over $K$, and $H^{2j-1}(X)(j)$ is the motive $H^{2j-1}(X) \otimes Q(j)$, where $Q(1) = H^2(P^1)^\vee$ and $Q(j) = Q(1)^{\otimes j}$. Also $Q(0)$ is the motive of a point.

More precisely, let $MHS$ be the category of (rational) mixed Hodge structures. One has the (mod torsion) Griffiths Abel-Jacobi map

$$AJ : \text{CH}^j(X)_{0,Q} \to JH^{2j-1}(X) \cong \text{Ext}^1_{MHS}(Q(0), H^{2j-1}(X)(j)),$$

where $JH^{2j-1}(X)$ is the intermediate Jacobian of Griffiths (modulo torsion) and the isomorphism is given by a theorem of Carlson [Ca80] on classification of extensions in $MHS$. This map is actually motivic. That is, there is a motivic Abel-Jacobi map

$$AJ_{\text{mot}} : \text{CH}^j(X)_{0,Q} \to \text{Ext}^1_{MHS}(Q(0), H^{2j-1}(X)(j)),$$

whose composition with the forgetful map

$$\text{Ext}^1_{MHS}(Q(0), H^{2j-1}(X)(j)) \to \text{Ext}^1_{MHS}(Q(0), H^{2j-1}(X)(j))$$

is the Abel-Jacobi of Griffiths. For an arbitrary field $K \subset \mathbb{C}$, Beilinson-Bloch conjectures predict $AJ_{\text{mot}}$ to be surjective. If $K$ is a number field, they expect $AJ_{\text{mot}}$ to be an isomorphism. Note that the Griffiths Abel-Jacobi map $AJ$ is also expected to be injective for varieties over number fields (this would follow from bijectivity of $AJ_{\text{mot}}$ and the Hodge-Nori conjecture).

The conjectures of Bloch and Beilinson go much beyond what is discussed above. See [Bl80], [Be87], [Sc91], [Ja94], and [Ne94] for more details.

3. WORK ON ALGEBRAIC CYCLES AND HODGE THEORY OF THE FUNDAMENTAL GROUP OF A PUNCTURED CURVE

Here we give a summary of [Es19]. All mixed Hodge structure in this section are integral. The Ext groups are all in the category of integral mixed Hodge structures.

Let $U$ be a smooth complex variety. Choose a base point $e$ in $U$. Let $I$ be the kernel of the map $\mathbb{Z}[\pi_1(U, e)] \to \mathbb{Z}$ given by $\gamma \mapsto 1$ (where $\pi_1(U, e)$ is the topological fundamental group, with $U$ considered as a complex manifold). We have

$$I \supset I^2 \supset \cdots \supset I^{n+1} \supset \cdots .$$

By Hurewicz theorem,

$$\left( \frac{I}{I^2} \right)^\vee \cong H_1(U, \mathbb{Z})^\vee \cong H^1(U, \mathbb{Z}),$$

Note that in Sections 1.1.1 and 3 we take the codomain of the Abel-Jacobi map to be the Griffiths intermediate Jacobian of a homology group. The passage between the two variants of the map is via Poincaré duality. Notationally, it is more convenient to work with the cohomological variant here (as we can then avoid introducing a parameter for the dimension of $X$).
so that \((1/1^2)\) underlies a mixed Hodge structure. Using the theory of K. T. Chen's iterated integrals, Hain ([Ha87a] and [Ha87b]) was able to show that each \((1/1^{n+1})\) underlies a mixed Hodge structure. This mixed Hodge structure, which we denote by \(L_n(U, e)\), is not necessarily a direct sum of pure Hodge structures even if \(U\) is projective. The elements of \(L_n(U, e)\) can be expressed as iterated integrals of length at most \(n\), explaining the notation \(L_n\). If \(n > 1\), the mixed Hodge structure \(L_n(U, e)\) depends on the base point \(e\).

The mixed Hodge structure \(L_n(U, e)\) is the Hodge realization of a mixed motive (over a subfield \(K \subset \mathbb{C}\), if \(U\) and \(e\) are defined over \(K\)). As such, Beilinson-Bloch conjectures predict that certain extensions that can be naturally constructed from \(L_n(U, e)\) (for suitable \(U\)) should be (after tensoring with \(\mathbb{Q}\)) in the image of Abel-Jacobi maps. This philosophy has been confirmed in certain situations. The first work in this spirit is by Harris [Ha83] and Pulte [Pu88]. From now on, let \(X\) be a smooth projective curve \(X\) over \(\mathbb{C}\). Pulte studies the extension

\[
0 \rightarrow W_1^{\text{L}_2}(X, e) \rightarrow W_2^{\text{L}_2}(X, e) \rightarrow \text{Gr}^W_2L_2(X, e) \rightarrow 0.
\]

He shows that, after suitable identifications, twice this extension (with respect to Baer sum) equals the Abel-Jacobi image of the Ceresa cycle \(X_0 - X^- \in CH^{\text{hom}}(\text{Jac})\), where \(\text{Jac}\) is the Jacobian of \(X\). (Here as usual, \(X_0\) is the image of \(X\) under the map \(X \rightarrow \text{Jac}\) given by \([x] \mapsto [x] - [e]\), and \(X^-\) is the pushforward of \(X_0\) under the inversion map.)

Later, Kaenders [Ka01] and Darmon, Rotger and Sols [DRS12] considered the analogous extension coming from \(L_2\) of a punctured curve. Building on the latter work, in [Es19] we consider an extension arising from \(L_n(X - \{\infty\}, e)\), where \(\infty\) is a point in \(X\). For simplicity, let us write \(H^1\) for \(H^1(X - \{\infty\}) = H^1(X)\) (identified via the map induced by the inclusion \(X - \{\infty\} \subset X\)). For \(m \leq n\), one has \(W_m^nL_n(X - \{\infty\}, e) = L_m(X - \{\infty\}, e)\), and moreover, there is an isomorphism \(L_n/L_{n-1}(X - \{\infty\}, e) \simeq (H^1)^{\otimes n}\), given by

\[
\int \omega_1 \ldots \omega_n + \text{lower length terms} \mapsto [\omega_1] \otimes \cdots \otimes [\omega_n]
\]

with the \(\omega_i\) closed smooth 1-forms on \(X - \{\infty\}\). Let \(E_n\) be the extension

\[
0 \rightarrow \text{Gr}^W_{n-1}L_n(X - \{\infty\}, e) \rightarrow \frac{W_n}{W_{n-2}L_n(X - \{\infty\}, e)} \rightarrow \text{Gr}^W_nL_n(X - \{\infty\}, e) \rightarrow 0
\]

considered as an element of the Yoneda Ext group (in the category of mixed Hodge structure)

\[
\text{Ext} \left( (H^1)^{\otimes n}, (H^1)^{\otimes n-1} \right) \overset{\text{Poincaré duality}}{\cong} \text{Ext} \left( (H^1)^{\otimes 2n-1}(n - 1), \mathbb{Z}(0) \right)
\]

(where \(M(n)\) is the twist by \(\mathbb{Z}(n)\)). By a theorem of Carlson on classifying extensions in the category of mixed Hodge structures [Ca80], the latter extension group can be identified as a Künneth direct summand in the Griffiths intermediate Jacobian \(JH_{2n-1}(X^{2n-1})\) associated to \(H_{2n-1}(X^{2n-1})\). We have an Abel-Jacobi map

\[
CH_{n-1}^{\text{hom}}(X^{2n-1}) \rightarrow JH_{2n-1}(X^{2n-1})
\]

The main result of [Es19] is the following:

**Theorem 1.** The extension \(E_n\) is the Abel-Jacobi image of an explicitly given element \(\Lambda_n \in CH_{n-1}^{\text{hom}}(X^{2n-1})\). The algebraic cycle \(\Lambda_n\) is defined over a subfield \(K \subset \mathbb{C}\) if \(X, \infty\) and \(e\) are defined over \(K\).
See the article for the construction of $\Lambda_n$. The result has the following application to number theory. For an integral Hodge class $\xi$, in $(H^1)^{\otimes 2n-2}$, or equivalently, a morphism of Hodge structures $\mathbb{Z}(1-n) \to (H^1)^{\otimes 2n-2}$, pulling back along $\xi$, gives a map

$$\xi^{-1} : \text{Ext} \left( (H^1)^{\otimes 2n-1}(n-1), \mathbb{Z}(0) \right) \to \text{Ext} \left( H^1, \mathbb{Z}(0) \right).$$

Using the theorem of Carlson and the classical Abel-Jacobi map, the extension group on the right can be identified with the complex points of the Jacobian of $X$. We show that if $X_n$, $e$ are defined over a subfield $K \subset \mathbb{C}$ and $\xi$ is the cohomology class of an algebraic cycle on $X^{2n-2}$ that is also defined over $K$, then the point $\xi^{-1}(\mathbb{P}_n)$ on the Jacobian of $X$ is $K$-rational. We should point out that for $n = 2$, this result is due to Darmon-Rotger-Sols [DRS12].

In the future I would like to explore whether this method can be useful for constructing nontorsion rational points on Jacobian varieties. In particular, I would be interested to explore whether working with $n > 2$ can result in any “new” rational points (i.e. points linearly independent to those appearing for $n = 2$).

4. Work on Mumford-Tate groups of mixed Hodge structures

Here we give a summary of [EM19b] and [EM19c], with remarks on some future directions. All mixed Hodge structures in this section are rational. We denote the category of mixed Hodge structures by $\mathcal{MHS}$. Given a mixed Hodge structure $M$, we denote by $\langle M \rangle$ the full Tannakian subcategory of $\mathcal{MHS}$ generated by $M$ (see [DM82] for the language of Tannakian categories$^4$). If $\omega$ is a fiber functor from $\mathcal{MHS}$ to the category of rational vector spaces, we denote by $G(M, \omega)$ the Tannakian group of the category $\langle M \rangle$ with respect to the fiber functor $\omega$. Thus $G(M, \omega)$ is an algebraic group over $\mathbb{Q}$ such that for any $\mathbb{Q}$-algebra $R$, the group of $R$-valued points $G(M, \omega)(R)$ is the group of automorphisms of the $\otimes$-functor $\omega \otimes 1_R$ from $\langle M \rangle$ to the category of $R$-modules. By the main theorem of the theory of Tannakian categories, one has an equivalence of categories from $\langle M \rangle$ to the category of finite-dimensional representations of $G(M, \omega)$ given by $N \mapsto \omega(N)$. The group $G(M, \omega)$ can be naturally identified with a closed subgroup of $\text{GL}(\omega(M))$ (by identifying $\sigma \in G(M, \omega)$ with its action on $\omega(M)$).

If $\omega = \omega_B$ is the forgetful (or Betti) fiber functor sending a mixed Hodge structure to its underlying rational vector space, $G(M, \omega)$ is called the Mumford-Tate group of $M$. In this case we shall drop the fiber functor from the notation and simply write $G(M)$. From an arithmetic point of view, the interest in the Mumford-Tate group comes from the following. If $M$ is a mixed motive over $K \subset \mathbb{C}$, similarly, considering the full Tannakian subcategory generated by $M$ in the category $\mathcal{MM}_K$ of mixed motives over $K$, we obtain the motivic Galois group of $M$. It is easy to see that the Mumford-Tate group is a subgroup of the motivic Galois group; if $K$ is algebraically closed, the Hodge-Nori conjecture predicts the two groups to be equal. On the other hand, when $K = \overline{\mathbb{Q}}$, Grothendieck’s period conjecture predicts the dimension of the motivic Galois group to be equal to the transcendence degree of the field generated by the periods of $M$.

The inclusion $\langle \text{Gr}^W M \rangle \subset \langle M \rangle$ gives a surjection $G(M, \omega) \to G(\text{Gr}^W M, \omega)$. Denote the kernel by $U(M, \omega)$. We will assume $M$ is polarizable. Then $U(M, \omega)$ is the unipotent radical of $G(M, \omega)$ and contains the “mixed data” of the category $\langle M \rangle$, i.e. the data of nontrivial extensions in this category. For $\omega = \omega_B$, again we simply write $U(M)$.

In a project joint with K. Murty, we consider the following problem:

**Problem 1.** Find a way to calculate $U(M)$ (or more generally $U(M, \omega)$).

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$^4$The definition of a Tannakian category over a field $F$ of characteristic zero is obtained by axiomatizing the properties enjoyed by the category of finite-representations of a group. Roughly speaking, it is an abelian category in which the Hom groups are vector spaces, compositions are linear, there is a tensor structure, object have duals and one has a canonical isomorphism $A \otimes V \cong A$ for every object $A$. Moreover, one requires existence of a so called fiber functor, i.e. a functor to the category of finite-dimensional $F$-vector spaces which is linear, faithful, exact, and respects the tensor structures.
4.1. In [EM19b] we consider the case where $M$ has two weights $a < b$. Choose a section for
\[ 0 \to \omega_B(Gr^WM_a) \to \omega_B(GM) \to \omega_B(Gr^WM_b) \to 0 \]
to identify $\omega_B(M) \cong \omega_B(Gr^WM_a) \oplus \omega_B(Gr^WM_b)$. The identification $G(M) \subset GL(\omega_B(M))$ identifies $\mathcal{U}(M)$ as a subgroup of
\[ \mathcal{U}_0 := \left\{ \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \in GL(\omega_B(Gr^WM_a) \oplus \omega_B(Gr^WM_b)) : f \in \text{Hom}(\omega_B(Gr^WM_b), \omega_B(Gr^WM_a)) \right\}, \]
so that
\[ \text{Lie}(\mathcal{U}(M)) \subset \text{Lie}(\mathcal{U}_0) \cong \text{Hom}(\omega_B(Gr^WM_b), \omega_B(Gr^WM_a)). \]
In view of the fundamental theorem of Tannakian categories, the adjoint representation of $G(M)$ induces a mixed Hodge structure $\text{Lie}(\mathcal{U}(M))$ with underlying rational vector space $\text{Lie}(\mathcal{U}(M))$. Since $\mathcal{U}(G)$ is abelian, this representation factors through a representation of $G(Gr^WM_b)$, so that the mixed Hodge structure $\text{Lie}(\mathcal{U}(M))$ belongs to $(Gr^WM)$. In fact, one easily sees by calculating the action of $G(M)$ that the inclusion $\text{Lie}(\mathcal{U}(M)) \subset \text{Hom}(\omega_B(Gr^WM_b), \omega_B(Gr^WM_a))$ is an inclusion of Hodge structures $\text{Lie}(\mathcal{U}(M)) \subset \text{Hom}(Gr^WM_b, Gr^WM_a)$, where $\text{Hom}$ is the internal Hom in $\mathcal{MHS}$.

For any subobject $V$ of $\text{Hom}(Gr^WM_a, Gr^WM_b)$, let $V^\perp$ be the orthogonal complement of $V$ with respect to the canonical nondegenerate pairing
\[ \text{Hom}(Gr^WM_a, Gr^WM_b) \otimes (Gr^WM_b)^\vee \otimes Gr^WM_a \to \mathbb{Q}(0). \]
One has a canonical isomorphism
\[ \text{Ext}(Gr^WM, Gr^WM_a) \cong \text{Ext}((Gr^WM_a)^\vee \otimes Gr^WM_b, \mathbb{Q}(0)) \]
(where $\text{Ext} = \text{Ext}_{\mathcal{MHS}}$). Let $\mu$ be the element of the group on the right corresponding to the extension $\text{Eq. (1)}$. The main result of [EM19b] is the following:

**Theorem 2.** Suppose $M$ is polarizable. Then $\text{Lie}(\mathcal{U}(M))^{\perp}$ is the largest subobject $S \subset (Gr^WM_a)^\vee \otimes Gr^WM_b$ such that $\mu |_S$ (= the restriction of $\mu$ to $S$) splits in $\mathcal{MHS}$.

In the article we then apply this to two particular examples:

(i) Let $X$ be a smooth projective curve over $\mathbb{C}$. We consider $\mathbb{L}_2(X, e)$ or $\mathbb{L}_2(X - \infty, e)$ in the notation of Section 3). Here the results of Pulte [Pu88] and Kaenders [Ka01] give us what we need to know about the splitting of restrictions of $\mu$.

(ii) We consider $H^1(X - S)$ with again $X$ a smooth projective curve and $S$ a finite set of points. In particular, we show that if $\text{End}(H^1(X)) = \mathbb{Q}$ (where $\text{End}(H^1(X))$ is the endomorphism algebra of the Hodge structure $H^1(X)$) and $g$ is the genus of $X$, then the dimension of $\mathcal{U}(H^1(X - S))$ equals $2g$ times the rank of the subgroup of the Jacobian of $X$ supported on $S$.

We should mention that Theorem 2 is also valid with $\mathcal{MHS}$ replaced with any Tannakian category of mixed motives or realizations in characteristic zero. Also one could work with any choice of fiber functor.

4.2. In an ongoing work with K. Murty we are trying to generalize the above to mixed Hodge structures (or mixed motives) with multiple weights. We give a report of the current state of the project here (details to be in [EM19c]). The first key step in [EM19b] is the following observation: writing $Gr^WM = Gr^WM_a \oplus Gr^WM_b$ with $a < b$, the Lie algebra $\text{Lie}(\mathcal{U}(M))$ can be naturally thought of as a Hodge substructure of the internal hom $\text{Hom}(Gr^WM_b, Gr^WM_a)$. The next step then is to identify this Hodge substructure, which is the subject of Theorem 2.

The first step can be generalized to arbitrary polarizable mixed Hodge structures (with possibly more than two weights), as follows. We change the fiber functor to the composition
\[ \omega : \mathcal{MHS} \xrightarrow{Gr^W} \mathcal{MHS}_{\text{forgetful}} \xrightarrow{\mathcal{Q}} \mathbb{Q}\text{-vector spaces}. \]
For an arbitrary polarizable mixed Hodge structure $M$, let us refer to the Tannakian group of $(M)$ with respect to the fiber functor $\omega$ as the graded Mumford-Tate group of $M$. By a theorem of Deligne [De90,
Theorem 1.12], Tannakian groups of a category with respect to different fiber functors are isomorphic after a finite base change, so that (say for applications to periods, where one would like to know the dimension of the Mumford-Tate group) one might as well work with the graded Mumford-Tate group $G(M, \omega)$. (In fact, $G(M, \omega)$ also describes the subcategory $\langle M \rangle$, so one is essentially not losing anything by changing the fiber functor.) We note that on the semi-simple category $G$ the two fiber functors $\omega$ and $\omega_B$ are canonically isomorphic, so that we can identify $G(\text{Gr}^W M, \omega) \cong G(\text{Gr}^W M)$. The key reason for working with the fiber functor $\omega$ is that the surjection $G(M, \omega) \to G(\text{Gr}^W M)$ admits a section, induced by $G^W : \langle M \rangle \to \langle \text{Gr}^W M \rangle$. (In particular, we have $G(M, \omega) \cong \text{U}(M, \omega) \times G(\text{Gr}^W M)$.)

Identifying $G(M, \omega)$ as a subgroup of $\text{GL}(\omega M) = \text{GL}(\oplus \omega_B \text{Gr}^W M)$, the subgroup $\text{U}(M, \omega)$ is contained in the subgroup

$$\{ f = (f_{\ell,n})_{\ell,n} : f_{\ell, n} \in \text{Hom}(\omega_B \text{Gr}^W_n M, \omega_B \text{Gr}^W_{\ell,n} M), f_{\ell,n} = 0 \text{ for } \ell > n, f_{\ell,\ell} = \text{Id} \} \subset \text{GL}(\omega M),$$

so that

$$\text{Lie}(\text{U}(M, \omega)) \subset \bigoplus_{\ell < n} \text{Hom}(\omega_B \text{Gr}^W_n M, \omega_B \text{Gr}^W_{\ell,n} M).$$

(The Lie algebra structure on the sum on right is given by thinking of the sum as the space of strictly upper block diagonal matrices in $\text{End}(\omega M)$. Equivalently, the Lie bracket on monomials is given by plus or minus the composition, and zero if the compositions do not make sense.) The adjoint action of $G(\text{Gr}^W M)$ on the Lie algebra $\text{Lie}(\text{U}(M, \omega))$ (which one has thanks to the section of the surjection $G(M, \omega) \to G(\text{Gr}^W M)$) induces a Hodge structure on $\text{Lie}(\text{U}(M, \omega))$. Calculating this action we see that the inclusion above is an inclusion of Hodge structures, i.e. $\text{Lie}(\text{U}(M, \omega))$ is a Hodge substructure of

$$\bigoplus_{\ell < n} \text{Hom}(\text{Gr}^W_n M, \text{Gr}^W_{\ell,n} M).$$

In order to describe this subobject, let $h$ be the highest weight of $M$. The inclusion $\langle W_{h-1} M \rangle \subset \langle M \rangle$ gives a surjection $\text{U}(M, \omega) \to \text{U}(W_{h-1} M, \omega)$; let $A$ be the kernel. Then one has a commutative diagram with exact rows

$$0 \to \text{Lie}(A) \longrightarrow \text{Lie}(\text{U}(M, \omega)) \longrightarrow \text{Lie}(\text{U}(W_{h-1} M, \omega)) \longrightarrow 0$$

$$0 \longrightarrow \bigoplus_{n < h} \text{Hom}(\omega_B \text{Gr}^W_n M, \omega_B \text{Gr}^W_{n,h} M) \longrightarrow \bigoplus_{\ell < n} \text{Hom}(\omega_B \text{Gr}^W_n M, \omega_B \text{Gr}^W_{\ell,n} M) \longrightarrow \bigoplus_{\ell < n, \ell < h} \text{Hom}(\omega_B \text{Gr}^W_n M, \omega_B \text{Gr}^W_{\ell,n} M) \longrightarrow 0,$$

The arrows on the second row are the inclusion and projection maps. All arrows and inclusions here are compatible with Hodge and Lie algebra structures. (In particular, $A$ is abelian.) The goal is to

(i) describe the subobject $\text{Lie}(A)$ of $\bigoplus_{n < h} \text{Hom}(\text{Gr}^W_n M, \text{Gr}^W_{n,h} M) = \text{Hom}(\text{Gr}^W_n M, \text{Gr}^W_{n,h} M)$, and

(ii) given $\text{Lie}(A)$ and $\text{Lie}(\text{U}(W_{h-1} M, \omega))$, find $\text{Lie}(\text{U}(M, \omega))$.

The following result accomplishes (i). Note that if $M$ has two weights, the result simplifies to the analogue of Theorem 2 for the fiber functor $\omega$.

**Theorem 3.** Let $\text{Lie}(A) \perp$ be the orthogonal complement of $\text{Lie}(A)$ with respect to the canonical pairing

$$\text{Hom}(\omega_B \text{Gr}^W_n M, \omega_B W_{h-1} M) \otimes (\omega_B W_{h-1} M)^\vee \otimes \omega_B \text{Gr}^W_M) \to \mathbb{Q}.$$ 

Let $\mu \in \text{Ext}(W_{h-1} M)^\vee \otimes \text{Gr}^W M, \mathbb{Q}(0))$ be the extension

$$0 \to \mathbb{Q}(0) \to M' \to (W_{h-1} M)^\vee \otimes \text{Gr}^W M \to 0$$

corresponding to

$$0 \to W_{h-1} M \to M \to \text{Gr}^W M \to 0$$

under the canonical isomorphism $\text{Ext}(\text{Gr}^W M, W_{h-1} M) \cong \text{Ext}((W_{h-1} M)^\vee \otimes \text{Gr}^W M, \mathbb{Q}(0))$. Let $N$ be the largest subobject of $M'$ which belongs to the subcategory $\langle W_{h-1} M \oplus \text{Gr}^W M \rangle$. Then $\text{Lie}(A) \perp = \omega_B(\pi(N))$. #
Again this result is valid in any Tannakian category of mixed motives or realizations. As for (ii), the case where the weights of \( \bigoplus_{0 \leq n \leq h} \text{Hom}(\text{Gr}_n^W M, \text{Gr}_1^W M) \) and \( \text{Hom}(\text{Gr}_n^W M, \text{Gr}^W_{h-1} M) \) are all distinct (which we call the separated weights case) is easy: strictness of morphisms of Hodge structures with respect to the weight together with semisimplicity of the category \( (\text{Gr}^W M) \) force the obvious section for the second row of Eq. (3) to restrict to a section for the first row, so that

\[
\text{Lie}(U(M, \omega)) \cong \text{Lie}(A) \times \text{Lie}(W_{h-1} M, \omega).
\]

4.3. We end this discussion with a few words on future directions. There are two avenues that we would like to explore in future work with K. Murty. One is to see what can be done in the case where the weights are unseparated. Another is to apply our method to some examples. Even more examples in the case of two weights would be interesting. For this we would like to look at extensions of mixed motives constructed by Deninger, Scholl and others.

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