THE FUNDAMENTAL GROUP OF AN EXTENSION IN A TANNAKIAN CATEGORY AND THE UNIPOTENT RADICAL OF THE MUMFORD-TATE GROUP OF AN OPEN CURVE

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Abstract. In the first part of the article, we give a self-contained account of Tannakian fundamental groups of extensions, generalizing a result of Hardouin from [16] and [15]. In the second part, we use Hardouin's characterization of Tannakian groups of extensions to give a characterization of the unipotent radical of the Mumford-Tate group of an open complex curve. Consequently, we prove a formula that relates the dimension of the unipotent radical of the Mumford-Tate group of an open complex curve $X \setminus S$ with X smooth and projective and S a finite set of points to the rank of the subgroup of the Jacobian of X supported on S.

1. Introduction

Let X be a smooth complex projective curve and $S \subset X(\mathbb{C})$ a finite nonempty set of points. There is an exact sequence

$$0 \longrightarrow H^1(X) \longrightarrow H^1(X \setminus S) \xrightarrow{\text{residue}} \mathbb{Q}(-1)^{|S|-1} \longrightarrow 0 \tag{1}$$

of (rational) mixed Hodge structures, where the first arrow is induced by the inclusion $X\setminus S \subset X$. In connection to a new proof of the Manin-Drinfeld theorem for modular curves, Deligne proved in the 1970s that this sequence splits (or equivalently, $H^1(X \setminus S)$ is semisimple) if and only if the rank of the subgroup of the Jacobian of X supported on S is zero (see [7, §10.3] and [8, Remarque 7.5], and also [11] for another argument).

To any mixed Hodge structure H, one associates an algebraic group called the Mumford-Tate group of H, which we denote by $\mathcal{MT}(H)$. This group can be defined in at least two equivalent ways: In the original definition, due to Mumford (and then refined by Serre) in the pure case, $\mathcal{MT}(H)$ is the subgroup of $GL(H_{\mathbb{Q}})$ (where as usual, $H_{\mathbb{Q}}$ denotes the underlying rational vector space of H) which fixes all Hodge classes of weight zero in finite direct sums of objects of the form

$$H^{\otimes m} \otimes (H^\vee)^{\otimes n} \qquad (m, n \in \mathbb{Z}_{\geq 0}).$$

The second definition, which is somewhat more natural and more conceptual, is in terms of Tannakian formalism: $\mathcal{MT}(H)$ is the fundamental group of the Tannakian subcategory $\langle H \rangle$ of the category of mixed Hodge structures generated by H (see Section 2 for a brief reminder on Tannakian fundamental groups; see [2] for the equivalence of the two definitions). This means that one has a canonical equivalence of categories between $\langle H \rangle$ and the category of finite-dimensional representations of $\mathcal{MT}(H)$.

The unipotent radical of $\mathcal{MT}(H)$ measures how far H is from being semisimple. In particular, H is semisimple if and only if the unipotent radical of $\mathcal{MT}(H)$ is trivial. Thus Deligne's result about $H^1(X \setminus S)$ can be paraphrased as follows: the unipotent radical of $\mathcal{MT}(H^1(X \setminus S))$ is trivial if and only if the rank of the subgroup of the Jacobian of X supported on S is zero.

The unipotent radical of the Mumford-Tate group of a 1-motive (of which the Mumford-Tate group of $H^1(X \setminus S)$ is an example) has been studied in great generality by Bertolin ([3] and [4]) and Jossen [17]. On his path to prove the main theorem of [17], Jossen gives a characterization of this unipotent radical in *loc. cit.*, Theorem 6.2.

In the case of $H^1(X \setminus S)$, Jossen's characterization is the following: Suppose $S = \{p_0, \ldots, p_n\}$. Let P be the connected component of the Zariski closure of the subgroup generated by

$$(p_1 - p_0, \, \ldots, \, p_n - p_0)$$

in $Jac(X)^n$, where $Jac(X)$ is the Jacobian of X. Then P itself is an abelian subvariety of $Jac(X)^n$. Jossen's theorem asserts that the Lie algebra of the unipotent radical of $\mathcal{MT}(H^1(X \setminus S))$ is canonically isomorphic to $H_1(P)$. In particular, the dimension of the unipotent radical of $\mathcal{MT}(H^1(X \setminus S))$ is twice the dimension of P.

To get a more concrete description (one that does not involve the Zariski closure) of the dimension of the unipotent radical of $\mathcal{MT}(H^1(X \setminus S))$, one can note that linear relations between the points $p_1 - p_0, \ldots, p_n - p_0$ with coefficients in the endomorphism algebra of $Jac(X)$ cut down the dimension of P.

One of the main results of this paper gives a more explicit description of the unipotent radical of $\mathcal{MT}(H^1(X \setminus S))$ that avoids the Zariski closure (see Theorem 4.8.1). As a consequence, in the simple case, we get the following clean formula for the dimension of the unipotent radical (see Theorem 4.8.2):

Theorem A. *Let X, S, and $Jac(X)$ be as above. Let g, E, and $\mathcal{U}(H^1(X \setminus S))$ be respectively the genus of X, the endomorphism algebra $End(Jac(X)) \otimes \mathbb{Q}$ of $Jac(X)$, and the unipotent radical of the Mumford-Tate group of $H^1(X \setminus S)$. Suppose that $Jac(X)$ is simple. Then the dimension of $\mathcal{U}(H^1(X \setminus S))$ is equal to $2g$ times the E-rank of the E-submodule of $Jac(X)(\mathbb{C}) \otimes \mathbb{Q}$ generated by the subgroup supported on S.*

Let us put this discussion on hold for the moment and go to the abstract setting of (neutral) Tannakian categories. Let $\mathbf{T}$ be a Tannakian category over a field K of characteristic zero, and ω a fiber functor over K (the example relevant to the earlier discussion being the category of mixed Hodge structures and the forgetful functor $H \mapsto H_{\mathbb{Q}}$). Suppose we have an extension

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{2}$$

in $\mathbf{T}$. Denoting the Tannakian fundamental groups of objects with respect to ω by $\mathcal{G}(-)$, we have a natural surjection

$$\mathcal{G}(M) \longrightarrow \mathcal{G}(L \oplus N).$$

Let $\mathcal{U}(M)$ be the kernel of this map (if N and L are semisimple, then $\mathcal{U}(M)$ is the unipotent radical of $\mathcal{G}(M)$). By Tannakian formalism, there is an object

$$\underline{Lie}(\mathcal{U}(M)) \subset \underline{Hom}(N, L)$$

whose image under ω is the Lie algebra of $\mathcal{U}(M)$. The question of characterization of $\underline{Lie}(\mathcal{U}(M))$ has been studied and answered earlier by Hardouin and Bertrand in the case where $N = \mathbb{1}$ and L is semisimple: A theorem of Hardouin ([16, Theorem 2], see also the unpublished work [15]) asserts that in this case, $\underline{Lie}(\mathcal{U}(M))$ is the smallest subobject of

$$\underline{Hom}(\mathbb{1}, L) \cong L$$

such that the pushforward of (2) along the quotient map

$$L \longrightarrow L/\underline{Lie}(\mathcal{U}(M))$$

splits. The result was earlier proved by Bertrand [5, Theorem 1.1] in the setting of D-modules.

The case of arbitrary semisimple N (with L continued to be semisimple as well) can be deduced from Hardouin's result. In this case, the characterization becomes as follows: If ν is the extension of $\mathbb{1}$ by $\underline{Hom}(N, L)$ corresponding to (2) under the canonical isomorphism

$$Ext(N, L) \;\cong\; Ext(\mathbb{1}, \underline{Hom}(N, L))$$

(where Ext means the Yoneda Ext^1 group in $\mathbf{T}$), then $\underline{Lie}(\mathcal{U}(M))$ is the smallest subobject of $\underline{Hom}(N, L)$ such that the pushforward of ν under the quotient map

$$\underline{Hom}(N, L) \;\longrightarrow\; \underline{Hom}(N, L)/\underline{Lie}(\mathcal{U}(M))$$

splits.

The goal of this paper is twofold. Our first goal, to which the first part of the paper is devoted, is to give a self-contained and general treatment of Tannakian groups of extensions in characteristic zero. More precisely, in the general setting of the extension (2) in a Tannakian category, in Theorem 3.3.1 we give a characterization of $\underline{Lie}(\mathcal{U}(M))$ as a subobject of $\underline{Hom}(N, L)$, without assuming that N or L is semisimple. In the semisimple case, the result simplifies to Hardouin's characterization (see Corollary 3.4.1). We also discuss a dual variant of the characterization of $\underline{Lie}(\mathcal{U}(M))$ (Theorem 3.5.1 and in the semisimple case, Corollary 3.5.2), which is more convenient in some settings.

We should point out that the generalization to the non-semisimple situation is indeed useful in practical applications: extensions as in (2) with non-semisimple L and N arise naturally, for example, in a non-semisimple Tannakian category with a weight filtration, e.g. the category of mixed motives. In fact, in [12] we build on Theorem 3.3.1 to refine a result of Deligne from [17, Appendix] on unipotent radicals of Tannakian fundamental groups in a Tannakian category with a weight filtration, and then give applications to mixed motives which have "large" unipotent radicals of motivic Galois groups (see the aforementioned paper for more details).

The second goal of the paper is to use results about Tannakian groups of extensions to study the unipotent radical of the Mumford-Tate group of an open curve, by taking $\mathbf{T}$ to be the category of mixed Hodge structures and applying the results to the extension (1). The main result (Theorem 4.8.1) gives a characterization of the unipotent radical of the Mumford-Tate group of an open curve, from which the dimension formula stated above follows (see Theorem 4.8.2). The proof of Theorem 4.8.1 has two ingredients: The first ingredient is the semisimple case of Theorem 3.3.1 due to Hardouin (or more precisely, its dual variant given in Corollary 3.5.2). This gives a characterization of $\underline{Lie}(\mathcal{U}(H^1(X \setminus S))$ as follows: if μ is the element of

$$Ext(H^1(X)^{|S|-1}, \mathbb{1})$$

corresponding to (1) under the canonical isomorphisms

$$Ext(\mathbb{Q}(-1)^{|S|-1}, H^1(X)) \;\cong\; Ext(H_1(X) \otimes \mathbb{Q}(-1)^{|S|-1}, \mathbb{1}) \;\overset{\text{Poincaré duality}}{\cong}\; Ext(H^1(X)^{|S|-1}, \mathbb{1}),$$

then the orthogonal complement (see Section 3.5) of $\underline{Lie}(\mathcal{U}(H^1(X \setminus S))$ is the largest subobject of $H^1(X)^{|S|-1}$ on which μ restricts to zero. The second ingredient of the argument is now the calculation of the restrictions of the extension μ along different maps $H^1(X) \longrightarrow H^1(X)^{|S|-1}$. We use methods from transcendental algebraic geometry to calculate these restrictions.

Theorem 4.8.1 can be deduced alternatively from Jossen's general characterization of the unipotent radical of the Mumford-Tate group of an arbitrary 1-motive given in [17, Theorem 6.2]. Although Theorem 4.8.1 is weaker than Jossen's [17, Theorem 6.2], we hope that the reader might find some value in the simplicity of our approach and exposition, which solely rely on the general material on Tannakian groups and the calculation of the relevant extensions in the category of mixed Hodge structures. This approach can be applied to any situation where the relevant extensions can be calculated and described nicely. It is also hopefully more accessible to some audiences.

The paper is organized as follows. In the next section, we recall some basic generalities about Tannakian categories. In Section 3 we prove the characterizations of $\underline{Lie}(\mathcal{U}(M))$ in a general Tannakian category and for general L and N (with notation as above). A reader not familiar with the language of Tannakian categories but familiar with properties of the category of mixed Hodge structures may assume in Sections 2 and 3 that $\mathbf{T}$ is the latter category and ω is the forgetful functor. In Section 4, we come back to the problem of studying the unipotent radical of the Mumford-Tate group of an open curve, and prove Theorems 4.8.1 and 4.8.2.

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2. Preliminaries

In this section we briefly recall a few facts and constructions about Tannakian categories. For any commutative ring R, let $\mathbf{Mod}_R$ denote the category of R-modules. Throughout, K is a field of characteristic zero. The categories of groups and commutative K-algebras are respective denoted by **Groups** and $\mathbf{Alg}_K$. For an affine group scheme $\mathcal{G}$ over K, let $\mathbf{Rep}(\mathcal{G})$ be the category of finite-dimensional representations of $\mathcal{G}$ over K. We use the language of [10] for the theory of Tannakian categories. Our Tannakian categories are all neutral.

2.1. Let $\mathbf{T}$ be a Tannakian category over K with unit object $\mathbb{1}$; thus $\mathbf{T}$ is a K-linear rigid abelian tensor category with the identity $\mathbb{1}$ of the tensor structure satisfying $End(\mathbb{1}) = K$, for which there exists a fiber functor, i.e. a K-linear exact faithful tensor functor

$$\mathbf{T} \longrightarrow \mathbf{Mod}_K.$$

Let ω be such a functor. Let

$$\underline{Aut}^{\otimes}(\omega) \ : \ \mathbf{Alg}_K \longrightarrow \mathbf{Groups}$$

be the functor that sends a commutative K-algebra R to

$$\begin{aligned} Aut^{\otimes}(\omega \otimes 1_R) \ := &\text{ the group of automorphisms of the functor} \\ &\ \omega \otimes 1_R : \mathbf{T} \longrightarrow \mathbf{Mod}_R \\ &\text{ respecting the tensor structures.} \end{aligned}$$

The fundamental theorem of the theory of Tannakian categories [10, Theorem 2.11] asserts that $\underline{Aut}^{\otimes}(\omega)$ is representable by an affine group scheme $\mathcal{G}(\mathbf{T},\omega)$ over K (so that $\underline{Aut}^{\otimes}(\omega)$ is the functor of points of $\mathcal{G}(\mathbf{T},\omega)$), and that the functor

$$\mathbf{T} \longrightarrow \mathbf{Rep}(\mathcal{G}(\mathbf{T},\omega))$$

sending

$$M \mapsto \omega M$$

(with the natural action of $\mathcal{G}(\mathbf{T},\omega)$ on ωM) is an equivalence of tensor categories. We call $\mathcal{G}(\mathbf{T},\omega)$ the fundamental (or the Tannakian) group of $\mathbf{T}$ with respect to ω.

If $\mathbf{T}'$ is also a Tannakian category over K, a tensor functor $\phi : \mathbf{T}' \longrightarrow \mathbf{T}$ gives rise to a morphism

$$\phi^{\#} \ : \ \mathcal{G}(\mathbf{T},\omega) \ \longrightarrow \ \mathcal{G}(\mathbf{T}',\omega\circ\phi)$$

of group schemes over K, sending an automorphism of $\omega\otimes 1_R$ for any K-algebra R to the obvious automorphism induced on $(\omega\otimes 1_R)\circ\phi = (\omega\circ\phi)\otimes 1_R$. The morphism $\phi^{\#}$ is surjective (or faithfully flat) if and only if ϕ is fully faithful and moreover, satisfies the following property: for every $M \in \mathbf{T}'$, every subobject of $\phi(M)$ is isomorphic to $\phi(L)$ for some subobject L of M (see [10, Proposition 2.21], for instance). In particular, if $\mathbf{T}'$ is a full Tannakian subcategory of $\mathbf{T}$ which is closed under taking subobjects, then the inclusion $\mathbf{T}' \subset \mathbf{T}$ gives rise to a surjective morphism $\mathcal{G}(\mathbf{T},\omega) \longrightarrow \ \mathcal{G}(\mathbf{T}',\omega|_{\mathbf{T}'})$, where $\omega|_{\mathbf{T}'}$ is the restriction of ω to $\mathbf{T}'$.

2.2. Let M be an object of $\mathbf{T}$. Let $\langle M\rangle$ denote the full Tannakian subcategory of $\mathbf{T}$ generated by M, that is, the smallest full Tannakian subcategory of $\mathbf{T}$ that contains M, and is closed under taking subobjects (or subquotients). Set

$$\mathcal{G}(M,\omega) \ := \ \mathcal{G}(\langle M\rangle,\omega|_{\langle M\rangle}) \ = \ \underline{Aut}^{\otimes}(\omega|_{\langle M\rangle});$$

we refer to this group as the fundamental (or the Tannakian) group of M with respect to ω. Starting with M and $\mathbb{1}$, we can obtain every object of $\langle M\rangle$ by finitely many iterations of taking direct sums, duals, tensor products, and subquotients. It follows that the natural map

$$\mathcal{G}(M,\omega) \ \longrightarrow \ GL_{\omega M} \qquad\qquad \sigma \ \mapsto \ \sigma_M$$

(restricting to the action on ωM) is injective, so that, indeed, $\mathcal{G}(M,\omega)$ is an algebraic group over K. (Here, complying with the standard notation for natural transformations, $\sigma_M : \omega M \longrightarrow \omega M$ is how σ acts on ωM.) Often we will identify $\mathcal{G}(M,\omega)$ as a subgroup of $GL_{\omega M}$ via the injection above.

Since $\langle M\rangle$ is closed under taking subobjects, the natural map $\mathcal{G}(\mathbf{T},\omega) \longrightarrow \mathcal{G}(M,\omega)$ (induced by the inclusion $\langle M\rangle \subset \mathbf{T}$) is surjective. The kernel of this map consists of all $\sigma \in \mathcal{G}(\mathbf{T},\omega)$ such that σ_M is identity (then by functoriality, σ_N is also identity for every $N \in \langle M\rangle$).

2.3. For any algebraic group G, let $Lie(G)$ be the Lie algebra of G. Let $\mathcal{N}$ be a normal subgroup of $\mathcal{G}(M,\omega)$. Consider the adjoint representation

$$Ad \ : \ \mathcal{G}(M,\omega) \ \longrightarrow \ GL_{Lie(\mathcal{N})}. \tag{3}$$

In view of the equivalence of categories

$$\langle M\rangle \ \longrightarrow \ \mathbf{Rep}(\mathcal{G}(M,\omega)) \qquad\qquad A \mapsto \omega A, \tag{4}$$

there is a canonical object $\underline{Lie}(\mathcal{N})$ in $\langle M\rangle$ with

$$\omega\underline{Lie}(\mathcal{N}) = Lie(\mathcal{N}),$$

such that the natural action of $\mathcal{G}(M,\omega)$ on $\omega\underline{Lie}(\mathcal{N})$ (through the definition of $\mathcal{G}(M,\omega)$ as the group of tensor automorphisms of the functor ω) coincides with the adjoint representation (3).

3. The fundamental group of an extension

The goal of this section is to study the fundamental group of an extension in a Tannakian category. As before, let $\mathbf{T}$ be a Tannakian category over a field K of characteristic zero. Fix a fiber functor $\omega : \mathbf{T} \longrightarrow \mathbf{Mod}_K$. We shall drop ω from the notation for fundamental groups, and simply write $\mathcal{G}(M)$ (for M an object of $\mathbf{T}$). We use the notation I_A for the identity map on an object A of a given category. We use an unadorned Hom to denote a Hom group in a category of modules, with the coefficient ring understood from the context.

3.1. Let L, M and N be objects of $\mathbf{T}$ given in an exact sequence

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{q} N \longrightarrow 0. \tag{5}$$

The inclusion $\iota : \langle L \oplus N \rangle \subset \langle M \rangle$ induces a surjective morphism

$$\iota^{\#} \; : \; \mathcal{G}(M) \longrightarrow \mathcal{G}(L \oplus N).$$

Let $\mathcal{U}(M)$ be the kernel of this map; it consists of those $\sigma \in \mathcal{G}(M)$ which act trivially on $\omega L \oplus \omega N$, or equivalently, on both ωL and ωN (i.e. $\sigma_L = I_{\omega L}$ and $\sigma_N = I_{\omega N}$). Note that while for simplicity we did not incorporate L and N in the notation for $\mathcal{U}(M)$, in general, $\mathcal{U}(M)$ will also depend on L and N. Our goal in this section is to study the group $\mathcal{U}(M)$.

First, let us describe the map $\iota^{\#}$ more concretely. Use the map i (see (5)) to identify ωL as a subspace of ωM. Moreover, once and for all, choose a section of the surjection $\omega q : \omega M \longrightarrow \omega N$ to identify

$$\omega M \; = \; \omega L \oplus \omega N$$

(as vector spaces). Then the functor ω applied to the sequence (5) gives

$$0 \longrightarrow \omega L \longrightarrow \omega L \oplus \omega N \longrightarrow \omega N \longrightarrow 0, \tag{6}$$

where the second and third arrows are the inclusion and projection maps.

Let σ be an element of $\mathcal{G}(M)$. Since σ is an automorphism of the functor ω, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \omega L & \longrightarrow & \omega L \oplus \omega N & \longrightarrow & \omega N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_L} & & \downarrow{\scriptstyle \sigma_M} & & \downarrow{\scriptstyle \sigma_N} & & \\
0 & \longrightarrow & \omega L & \longrightarrow & \omega L \oplus \omega N & \longrightarrow & \omega N & \longrightarrow & 0.
\end{array}$$

It follows that

$$\sigma_M = \begin{pmatrix} \sigma_L & f \\ 0 & \sigma_N \end{pmatrix} \in GL_{\omega L \oplus \omega N}$$

for some $f \in Hom(\omega N, \omega L)$. Let

$$\mathcal{G}_0(M) \; \subset \; GL_{\omega L \oplus \omega N}$$

be the subgroup consisting of the elements which stabilize ωL. Regarding $\mathcal{G}(M)$ as a subgroup of $GL_{\omega M} = GL_{\omega L \oplus \omega N}$ (via $\sigma \mapsto \sigma_M$), we have

$$\mathcal{G}(M) \; \subset \; \mathcal{G}_0(M).$$

Similarly, for any σ in $\mathcal{G}(L \oplus N)$,

$$\sigma_{L \oplus N} = \begin{pmatrix} \sigma_L & 0 \\ 0 & \sigma_N \end{pmatrix} \in GL_{\omega L \oplus \omega N}.$$

Thinking of $\mathcal{G}(L\oplus N)$ (resp. $GL_{\omega L}\times GL_{\omega N}$) as a subgroup of $GL_{\omega L\oplus\omega N}$ via $\sigma\mapsto\sigma_{L\oplus N}$ (resp. the diagonal embedding), we have

$$\mathcal{G}(L\oplus N)\ \subset\ GL_{\omega L}\times GL_{\omega N}.$$

The map $\iota^{\#}$ is then the restriction of

$$\begin{aligned}\varphi\ :\ \mathcal{G}_0(M)\ &\longrightarrow\ GL_{\omega L}\times GL_{\omega N}\\ \begin{pmatrix} g & * \\ 0 & g'\end{pmatrix}\ &\mapsto\ \begin{pmatrix} g & 0 \\ 0 & g'\end{pmatrix} \qquad\qquad (g\in GL_{\omega L},\ g'\in GL_{\omega N}).\end{aligned}$$

Let

$$\mathcal{U}_0(M)\ :=\ \ker(\varphi).$$

Thus $\mathcal{U}_0(M)$ is the subgroup of $GL_{\omega L\oplus\omega N}$ consisting of the elements of the form

$$\begin{pmatrix} I_{\omega L} & * \\ 0 & I_{\omega N}\end{pmatrix},$$

and in particular, is an abelian unipotent group. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{U}(M) & \longrightarrow & \mathcal{G}(M) & \xrightarrow{\iota^{\#}} & \mathcal{G}(L\oplus N) & \longrightarrow & 1\\ & & & & \cap & & \cap & & \\ 1 & \longrightarrow & \mathcal{U}_0(M) & \longrightarrow & \mathcal{G}_0(M) & \xrightarrow{\varphi} & GL_{\omega L}\times GL_{\omega N} & \longrightarrow & 1,\end{array} \tag{7}$$

where the injective arrows are inclusion maps and the rows are exact. Thus

$$\mathcal{U}(M)\ \subset\ \mathcal{U}_0(M).$$

In particular, $\mathcal{U}(M)$ is an abelian unipotent group.

As discussed in Section 2.3, the adjoint representation of $\mathcal{G}(M)$ gives a canonical object $\underline{Lie}(\mathcal{U}(M))$ of $\langle M\rangle$ whose image under ω is $Lie(\mathcal{U}(M))$. Since $\mathcal{U}(M)$ is abelian, the action of $\mathcal{G}(M)$ on $Lie(\mathcal{U}(M))$ factors through an action of $\mathcal{G}(L\oplus N)$, so that indeed, the object $\underline{Lie}(\mathcal{U}(M))$ belongs to the subcategory $\langle L\oplus N\rangle$.

The Lie algebra of $\mathcal{U}_0(M)$ can be identified with

$$Hom(\omega N,\omega L)$$

(with trivial Lie bracket). The exponential map

$$\exp\ :\ Lie(\mathcal{U}_0(M)) = Hom(\omega N,\omega L)\ \longrightarrow\ \mathcal{U}_0(M)(K)$$

is given by

$$\exp(f)\ =\ \begin{pmatrix} I_{\omega L} & f \\ 0 & I_{\omega N}\end{pmatrix}. \tag{8}$$

Let $\underline{Hom}(N,L)$ denote the internal hom object in the category $\mathbf{T}$. We identify

$$\omega(\underline{Hom}(N,L))\ =\ Hom(\omega N,\omega L)$$

via the canonical isomorphism between the two.

The following observation is standard.

Proposition 3.1.1. *The inclusion map*

$$Lie(\mathcal{U}(M)) \longrightarrow Hom(\omega N, \omega L)$$

is ω of a morphism

$$\underline{Lie}(\mathcal{U}(M)) \longrightarrow \underline{Hom}(N, L).$$

(In other words, $\underline{Lie}(\mathcal{U}(M))$ can be identified as a subobject of $\underline{Hom}(N, L)$.)

Proof. In view of the equivalence of categories (4), it is enough to show that the natural actions of $\mathcal{G}(M)$ on $Lie(\mathcal{U}(M))$ and $Hom(\omega N, \omega L)$ are compatible. In other words, we need to show that for any commutative K-algebra R and $\sigma \in \mathcal{G}(M)(R)$, we have

$$\sigma_{\underline{Lie}(\mathcal{U}(M))} = \sigma_{\underline{Hom}(N,L)}|_{Lie(\mathcal{U}(M))_R},$$

where for any vector space V over K, we denote $V_R := V \otimes R$. We may identify

$$(\omega \underline{Hom}(N, L))_R = Hom((\omega N)_R, (\omega L)_R))$$

(Hom in R-modules). Considering the evaluation map $N \otimes N^\vee \longrightarrow \mathbb{1}$ and the canonical isomorphism $\underline{Hom}(N, L) \cong N^\vee \otimes L$ (which after applying ω, are the corresponding maps in linear algebra), one easily sees that the map $\sigma_{\underline{Hom}(N,L)}$ is given by

$$f \mapsto \sigma_L \circ f \circ \sigma_N^{-1} \qquad (f \in Hom((\omega N)_R, (\omega L)_R)).$$

We now calculate the map $\sigma_{\underline{Lie}(\mathcal{U}(M))}$. By definition, the action of $\mathcal{G}(M)$ on $Lie(\mathcal{U}(M))$ is the restriction of the adjoint representation of $\mathcal{G}(M)$ to $Lie(\mathcal{U}(M))$. Let

$$f \in Lie(\mathcal{U}(M))_R \subset Lie(\mathcal{U}_0(M))_R = Hom((\omega N)_R, (\omega L)_R).$$

Then $\sigma_{\underline{Lie}(\mathcal{U}(M))}(f)$ is characterized by

$$\exp(\sigma_{\underline{Lie}(\mathcal{U}(M))}(f)) = \sigma_M \exp(f) \sigma_M^{-1},$$

where exp is the isomorphism between $Lie(\mathcal{U}(M))$ and $\mathcal{U}(M)$ as varieties over K, and via the inclusion $\mathcal{U}(M) \subset \mathcal{U}_0(M)$, is given by (8) (with coefficients extended to R). Writing

$$\sigma_M = \begin{pmatrix} \sigma_L & h \\ 0 & \sigma_N \end{pmatrix}$$

where $h \in Hom((\omega N)_R, (\omega L)_R)$, we have

$$\begin{aligned} \sigma_M \exp(f) \sigma_M^{-1} &= \begin{pmatrix} \sigma_L & h \\ 0 & \sigma_N \end{pmatrix} \begin{pmatrix} I_{(\omega L)_R} & f \\ 0 & I_{(\omega N)_R} \end{pmatrix} \begin{pmatrix} \sigma_L^{-1} & -\sigma_L^{-1} \circ h \circ \sigma_N^{-1} \\ 0 & \sigma_N^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I_{(\omega L)_R} & \sigma_L \circ f \circ \sigma_N^{-1} \\ 0 & I_{(\omega N)_R} \end{pmatrix} \\ &= \exp(\sigma_L \circ f \circ \sigma_N^{-1}). \end{aligned}$$

Thus

$$\sigma_{\underline{Lie}(\mathcal{U}(M))}(f) = \sigma_L \circ f \circ \sigma_N^{-1},$$

as desired. ∎

Remark 1.

(1) The embedding
$$Lie(\mathcal{U}(M)) \ \subset \ Hom(\omega N, \omega L)$$
is independent of the section of ωq used to identify $\omega M = \omega L \oplus \omega N$. Indeed, if we had chosen a different section of ωq and hence a different identification of ωM as $\omega L \oplus \omega N$, then the resulting embedding $\mathcal{G}(M) \hookrightarrow GL_{\omega L \oplus \omega N}$ would differ from the previous one by conjugation by an element of $\mathcal{U}_0(M)$. Since $\mathcal{U}_0(M)$ is abelian, the two embeddings agree on $\mathcal{U}(M)$. Thus our identification of $\underline{Lie}(\mathcal{U}(M))$ as a subobject of $\underline{Hom}(N, L)$ is independent of the choice of the section of ωq.

(2) If L and N are semisimple, then $\mathcal{U}(M)$ is the unipotent radical of $\mathcal{G}(M)$, and in particular will only depend on M (and not on the choices of L or N). (Recall that L and N are semisimple if and only if the category $\langle L \oplus N \rangle$ is semisimple if and only if $\mathcal{G}(L \oplus N)$ is reductive.)

3.2. Before we proceed any further, let us recall a categorical construction. The extension (5) gives an element of
$$Ext(N, L),$$
where Ext denotes the Yoneda Ext^1 group in $\mathbf{T}$. Recall that one has a canonical isomorphism
$$Ext(N, L) \ \cong \ Ext(\mathbb{1}, \underline{Hom}(N, L)). \tag{9}$$
Let
$$\nu \ \in \ Ext(\mathbb{1}, \underline{Hom}(N, L))$$
be the extension class corresponding to (5) under the canonical isomorphism (9). The extension ν is obtained by first applying $\underline{Hom}(N, -)$ to the sequence (5):
$$0 \ \longrightarrow \ \underline{Hom}(N, L) \ \longrightarrow \ \underline{Hom}(N, M) \ \longrightarrow \ \underline{Hom}(N, N) \ \longrightarrow \ 0,$$
and then pulling back along the canonical morphism
$$e \ : \ \mathbb{1} \ \longrightarrow \ \underline{Hom}(N, N),$$
characterized by that
$$\omega e(1) \ \in \ \omega \underline{Hom}(N, N) \ = \ Hom(\omega N, \omega N)$$
is the identity map. Going through this procedure, assuming $N \neq 0$, we see that ν is the class of the extension
$$0 \ \longrightarrow \ \underline{Hom}(N, L) \ \longrightarrow \ \underline{Hom}(N, M)^{\dagger} \ \longrightarrow \ \mathbb{1} \ \longrightarrow \ 0, \tag{10}$$
where

- $\underline{Hom}(N, M)^{\dagger}$ is the subobject of $\underline{Hom}(N, M)$ characterized by
$$\begin{aligned} \omega \underline{Hom}(N, M)^{\dagger} \ &= \ Hom(\omega N, \omega M)^{\dagger} \\ &:= \Big\{ \ f \in Hom(\omega N, \omega M) : (\omega q) \circ f = \lambda(f) Id_{\omega N} \text{ for some } \lambda(f) \in K \ \Big\}, \end{aligned}$$
- after applying ω, the injective arrow is $f \mapsto (\omega i) \circ f$, and
- after applying ω, the surjective arrow is the map $f \mapsto \lambda(f)$, where $\lambda(f) \in K$ is as in the definition of $\underline{Hom}(N, M)^{\dagger}$ above.

If N (and hence $\underline{Hom}(N, L)$) is zero , then ν is the trivial extension
$$0 \ \longrightarrow \ 0 \ \longrightarrow \ \mathbb{1} \ \longrightarrow \ \mathbb{1} \ \longrightarrow \ 0.$$
For convenience, we set $\underline{Hom}(N, M)^{\dagger} := \mathbb{1}$ in this case.

3.3. We are ready to give the characterization of the subobject $\underline{Lie}(\mathcal{U}(M))$ of $\underline{Hom}(N,L)$. To simplify the notation, we identify $\underline{Hom}(N,L)$ with its image under the injection $\underline{Hom}(N,L) \longrightarrow \underline{Hom}(N,M)^\dagger$.

Theorem 3.3.1. *Let A be a subobject of $\underline{Hom}(N,L)$. Then A contains $\underline{Lie}(\mathcal{U}(M))$ if and only if the quotient*

$$\underline{Hom}(N,M)^\dagger / \, A$$

belongs to the subcategory $\langle L\oplus N\rangle$. (Thus $\underline{Lie}(\mathcal{U}(M))$ is the smallest subobject of $\underline{Hom}(N,L)$ with this property.)

Proof. The theorem is trivial if $N=0$, so we may assume $N\neq 0$. An object X of $\langle M\rangle$ belongs to the subcategory $\langle L\oplus N\rangle$ if and only if the subgroup $\mathcal{U}(M)$ of $\mathcal{G}(M)$ acts trivially on $\omega\, X$. Thus the assertion in the theorem can be paraphrased as that A contains $\underline{Lie}(\mathcal{U}(M))$ if and only if the action of $\mathcal{U}(M)$ on $\omega(\underline{Hom}(N,M)^\dagger/A)$ is trivial.

Let $\sigma\in\mathcal{G}(M)(K)$. Let $A\subset \underline{Hom}(N,L)$. The morphism

$$\underline{Hom}(N,M)^\dagger \;\longrightarrow\; \underline{Hom}(N,M)^\dagger/A$$

gives rise to a commutative diagram

$$\begin{array}{ccc} Hom(\omega N,\omega M)^\dagger & \longrightarrow & Hom(\omega N,\omega M)^\dagger/\omega A \\ \Big\downarrow {\scriptstyle \sigma_{\underline{Hom}(N,M)^\dagger}} & & \Big\downarrow {\scriptstyle \sigma_{\underline{Hom}(N,M)^\dagger/A}} \\ Hom(\omega N,\omega M)^\dagger & \longrightarrow & Hom(\omega N,\omega M)^\dagger/\omega A\ . \end{array}$$

Thus

$$\sigma_{\underline{Hom}(N,M)^\dagger/A}(f+\omega A) \;=\; \sigma_{\underline{Hom}(N,M)^\dagger}(f)+\omega A$$

for every $f\in Hom(\omega N,\omega M)^\dagger$.

As before, we use our fixed section of $\omega q:\omega M\longrightarrow \omega N$ to identify $\omega M=\omega L\oplus\omega N$. Then we have

$$\begin{array}{ccc} Hom(\omega N,\omega M) & = & Hom(\omega N,\omega L)\oplus Hom(\omega N,\omega N) \\ \bigcup & & \bigcup \\ Hom(\omega N,\omega M)^\dagger & = & Hom(\omega N,\omega L)\oplus K\cdot I_{\omega N}. \end{array} \tag{11}$$

Suppose $\sigma\in\mathcal{U}(M)(K)$. Then σ_L and σ_N are both identity, and the action of σ on the $\mathcal{G}(M)$ - invariant subspace $Hom(\omega N,\omega L)$ of $Hom(\omega N,\omega M)^\dagger$ is trivial. Thus

$$\sigma_{\underline{Hom}(N,M)^\dagger/A} \;=\; I_{Hom(\omega N,\omega M)^\dagger/\omega A}$$

if and only if

$$\sigma_{\underline{Hom}(N,M)^\dagger/A}(I_{\omega N}+\omega A) \;=\; I_{\omega N}+\omega A$$

where here, as well as in the rest of this argument except in (12) below, $I_{\omega N}$ is considered as an element of $Hom(\omega N,\omega M)^\dagger$ via the decomposition (11). This is equivalent to

$$\sigma_{\underline{Hom}(N,M)^\dagger}(I_{\omega N})-I_{\omega N}\;\in\;\omega A.$$

Note that $\sigma_{\underline{Hom}(N,M)}$ (and hence $\sigma_{\underline{Hom}(N,M)^\dagger}$) is given by the formula

$$f\;\mapsto\;\sigma_M\circ f\circ\sigma_N^{-1}=\sigma_M\circ f \qquad\qquad (f\in Hom(\omega N,\omega M)).$$

We have

$$\sigma_M \;=\; \begin{pmatrix} I_{\omega L} & \log(\sigma_M) \\ 0 & I_{\omega N} \end{pmatrix} \;\in\; GL_{\omega L \oplus \omega N}(K), \tag{12}$$

where $\log(\sigma_M) \in Hom(\omega N, \omega L)$. Then

$$\sigma_{\underline{Hom}(N,M)^\dagger}(I_{\omega N}) \;=\; \sigma_M \circ I_{\omega N} \;=\; \log(\sigma_M) + I_{\omega N},$$

so that

$$\sigma_{\underline{Hom}(N,M)^\dagger}(I_{\omega N}) - I_{\omega N} = \log(\sigma_M).$$

We have shown that any element $\sigma \in \mathcal{U}(M)(K)$ acts trivially on $\omega(\underline{Hom}(N,M)^\dagger/A)$ if and only if $\log(\sigma_M)$ is in ωA. The group $\mathcal{U}(M)$ is unipotent and hence $\mathcal{U}(M)(K)$ is dense in $\mathcal{U}(M)$. It follows that $\mathcal{U}(M)$ acts trivially on $\omega(\underline{Hom}(N,M)^\dagger/A)$ if and only if for every $\sigma \in \mathcal{U}(M)(K)$, we have $\log(\sigma_M) \in \omega A$, i.e. if and only if $Lie(\mathcal{U}(M) \subset \omega A$. This completes the proof. □

3.4. For every subobject A of $\underline{Hom}(N,L)$, pushing extensions forward along the natural map $\underline{Hom}(N,L) \longrightarrow \underline{Hom}(N,L)/A$ we have a map

$$Ext(\mathbb{1}, \underline{Hom}(N,L)) \;\longrightarrow\; Ext(\mathbb{1}, \underline{Hom}(N,L)/A).$$

We denote the image of ν under this map by ν/A. Theorem 3.3.1 has the following corollary:

Corollary 3.4.1.

(a) *If A is a subobject of $\underline{Hom}(N,L)$ such that ν/A is trivial, then $\underline{Lie}(\mathcal{U}(M)) \subset A$.*
(b) *Suppose L and N are semisimple. Then $\nu/\underline{Lie}(\mathcal{U}(M))$ is trivial (and hence $\underline{Lie}(\mathcal{U}(M))$ is the smallest subobject of $\underline{Hom}(N,L)$ with this property).*

Proof. We may assume $N \neq 0$. Suppose ν/A is trivial. Then

$$\underline{Hom}(N,M)^\dagger/A \;\simeq\; \underline{Hom}(N,L)/A \;\oplus\; \mathbb{1},$$

and hence $\underline{Hom}(N,M)^\dagger/A$ belongs to the subcategory generated by L and N. Thus (a) follows from Theorem 3.3.1.

As for (b), the theorem implies that $\underline{Hom}(N,M)^\dagger/\underline{Lie}(\mathcal{U}(M))$ is in $\langle L \oplus N\rangle$, which is a semisimple category by the hypothesis of semisimplicity of L and N. Thus $\nu/\underline{Lie}(\mathcal{U}(M))$ splits. □

Remark 2. Corollary 3.4.1 is originally due to Hardouin (see [16] as well as the related unpublished article [15]). (Hardouin takes $N = \mathbb{1}$, but one can deduce the case of arbitrary N from that.) In the special case where $\mathbf{T}$ is the category of D-modules over a differential field of characteristic zero the result was earlier proved by Bertrand ([5, Theorem 1.1]).

3.5. Consider the canonical nondegenerate pairing

$$(L^\vee \otimes N) \;\otimes\; \underline{Hom}(N,L) \;\longrightarrow\; \mathbb{1} \tag{13}$$

given (after applying ω) by

$$(\gamma \otimes x) \otimes f \;\mapsto\; \gamma(f(x)).$$

For any subobject A of $\underline{Hom}(N,L)$ (resp. $L^\vee \otimes N$), we denote by $A^\perp$ the subobject of $L^\vee \otimes N$ (resp. $\underline{Hom}(N,L)$) orthogonal to A with respect to the above pairing. It is clear that A can be recovered from $A^\perp$ by $A^{\perp\perp} = A$.

In particular, we have a subobject

$$\underline{Lie}(\mathcal{U}(M))^\perp \;\subset\; L^\vee \otimes N.$$

In this subsection we shall give a dual variant of Theorem 3.3.1 which characterizes this object. In some situations (such as the application in Section 4), this variant might be more convenient to use than the original version.

Let

$$\mu \ \in \ Ext(L^\vee \otimes N, \mathbb{1})$$

be the extension class corresponding to the defining extension of M (i.e. (5)) under the canonical isomorphism

$$Ext(N, L) \ \cong \ Ext(L^\vee \otimes N, \mathbb{1}). \tag{14}$$

The extension μ is obtained as follows. Let

$$ev : L^\vee \otimes L \ \longrightarrow \ \mathbb{1}$$

be the evaluation pairing between L and its dual. Then μ is the pushforward of the extension

$$0 \ \longrightarrow \ L^\vee \otimes L \ \xrightarrow{I_{L^\vee} \otimes i} \ L^\vee \otimes M \ \xrightarrow{I_{L^\vee} \otimes q} \ L^\vee \otimes N \ \longrightarrow \ 0$$

(obtained by tensoring (5) by $L^\vee$) through the morphism ev. More explicitly, when L is not zero, μ is the extension

$$0 \ \longrightarrow \ \mathbb{1} \ \longrightarrow \ (L^\vee \otimes M)^\dagger \longrightarrow \ L^\vee \otimes N \ \longrightarrow \ 0, \tag{15}$$

where

- $(L^\vee \otimes M)^\dagger$ is the quotient of $L^\vee \otimes M$ by $(I_{L^\vee} \otimes i)(\ker(ev))$,
- the injective arrow is the composition

$$\mathbb{1} \ \xrightarrow{\simeq,\text{ induced by } ev} \ (L^\vee \otimes L) \ / \ \ker(ev) \ \xrightarrow{\text{induced by } I_{L^\vee} \otimes i} \ (L^\vee \otimes M)^\dagger,$$

  and
- the surjective arrow is induced by $I_{L^\vee} \otimes q$.

If $L = 0$, then μ is the extension

$$0 \ \longrightarrow \ \mathbb{1} \ \longrightarrow \ \mathbb{1} \ \longrightarrow \ 0 \ \longrightarrow \ 0.$$

For convenience, in this case we set $(L^\vee \otimes M)^\dagger \ := \ \mathbb{1}$.

We shall use the following notation for restrictions of extensions. For every subobject B of $L^\vee \otimes N$, let $\mu|_B$ be the restriction of μ to B (i.e. the pullback of μ along the inclusion map $B \longrightarrow L^\vee \otimes N$).

We can now state the dual variants of Theorem 3.3.1 and Corollary 3.4.1.

Theorem 3.5.1. *Let B be a subobject of $L^\vee \otimes N$. Then*

$$B \subset \underline{Lie}(\mathcal{U}(M))^\perp$$

if and only if the preimage of B under the surjective arrow in (15) *belongs to the subcategory $\langle L \oplus N \rangle$.*

Proof. One can prove this directly, similar to the proof of Theorem 3.3.1, by calculating the action of $\mathcal{U}(M)$ on $(L^\vee \otimes M)^\dagger$ (and its subobjects) explicitly. We shall instead use a few categorical considerations to show that the statement is equivalent to Theorem 3.3.1. Let T be an object of $\mathbf{T}$. For any subobject A of T, denote by $A^\perp$ the orthogonal complement of A with respect to the evaluation pairing

$$T^\vee \otimes T \ \longrightarrow \ \mathbb{1}.$$

Dualizing the exact sequence

$$0 \longrightarrow A \longrightarrow T \longrightarrow T/A \longrightarrow 0,$$

we get

$$0 \longrightarrow (T/A)^\vee \longrightarrow T^\vee \longrightarrow A^\vee \longrightarrow 0.$$

Use this to identify

$$A^\perp \overset{\text{(by definition)}}{=} \ker(T^\vee \longrightarrow A^\vee) \cong (T/A)^\vee.$$

There is a commutative diagram

$$\begin{tikzcd} Ext(\mathbb{1}, T) \arrow[r, "\text{dualizing, } \simeq"] \arrow[d, "\text{pushforward}"'] & Ext(T^\vee, \mathbb{1}) \arrow[d, "\text{pullback}"] \\ Ext(\mathbb{1}, T/A) \arrow[r, "\text{dualizing, } \simeq"] & Ext(A^\perp, \mathbb{1}), \end{tikzcd}$$

where the horizontal maps dualize extensions. Apply this with $T = \underline{Hom}(N, L)$, and use the pairing (13) to identify $L^\vee \otimes N$ as $T^\vee$ (so that (13) becomes simply the evaluation pairing between T and $T^\vee$). It is easy to see that ν and μ are duals of one another, with the isomorphism between $(L^\vee \otimes M)^\dagger$ and the dual of $\underline{Hom}(N, M)^\dagger$ defined by the pairing

$$(L^\vee \otimes M)^\dagger \otimes \underline{Hom}(N, M)^\dagger \longrightarrow \mathbb{1}$$

which after applying ω is given by

$$\overline{g \otimes x} \otimes f \mapsto g(\lambda(f)x - f((\omega q)(x))).$$

(Here $\overline{g \otimes x}$ is the image of $g \otimes x \in \omega(L^\vee) \otimes \omega M$ in $\omega(L^\vee \otimes M)^\dagger$, and f is in $Hom(\omega N, \omega M)^\dagger$.) Thus by the above diagram, for any subobject A of $\underline{Hom}(N, L)$, we have an isomorphism between

$$(\underline{Hom}(N, M)^\dagger / A)^\vee$$

and the preimage of $A^\perp$ under the surjective arrow in (15). The equivalence of Theorems 3.3.1 and 3.5.1 is clear from this. □

The argument also gives the following dual variant of Corollary 3.4.1:

Corollary 3.5.2.

(a) *If B is a subobject of $L^\vee \otimes N$ such that $\mu|_B$ is trivial, then $B \subset \underline{Lie}(\mathcal{U}(M))^\perp$.*
(b) *Suppose L and N are semisimple. Then the restriction of μ to $\underline{Lie}(\mathcal{U}(M))^\perp$ is trivial. (Hence $\underline{Lie}(\mathcal{U}(M))^\perp$ is the largest subobject of $L^\vee \otimes N$ with this property.)*

4. The unipotent radical of the Mumford-Tate group of H^1 of an algebraic curve

Let **MHS** be the category of rational mixed Hodge structures. The category **MHS** is a neutral Tannakian category over $\mathbb{Q}$. The forgetful functor $\omega_B : \mathbf{MHS} \to \mathbf{Mod}_{\mathbb{Q}}$ (sending an object to its underlying rational vector space) is a fiber functor. For any rational mixed Hodge structure M, the group $\mathcal{G}(M)$ with $(\mathbf{T}, \omega) = (\mathbf{MHS}, \omega_B)$ is called the Mumford-Tate group of M. In this section, we will use the results of the previous section to study the unipotent radical of the Mumford-Tate group of the degree one cohomology of a smooth complex projective curve minus a finite set of points.

4.1. Notation. By a mixed Hodge structure we always mean a rational one. By $\mathbb{Q}(-n)$ we denote the unique Hodge structure of weight $2n$ with underlying rational vector space $\mathbb{Q}$ (so that $\mathbb{Q}(0) = \mathbb{1}$ is the unit object). For any object M of **MHS**, we denote by $M_{\mathbb{Q}}$ the underlying rational vector space. If R is a commutative $\mathbb{Q}$-algebra, M_R denotes $M_{\mathbb{Q}} \otimes R$.

Given a pure Hodge structure H of odd weight $2k-1$, we denote by JH the intermediate Jacobian

$$JH := \frac{H_{\mathbb{C}}}{F^k H_{\mathbb{C}} + H_{\mathbb{Q}}}.$$

Given any smooth complex variety X, by $H^i(X)$ we mean the mixed Hodge structure on the degree i Betti cohomology of X (with underlying rational vector space $H^i(X,\mathbb{Q})$). We shall identify $H^i(X)_{\mathbb{C}} = H^i(X,\mathbb{C})$ with $H^i_{dR}(X)$ (= smooth complex de Rham cohomology) via the isomorphism of de Rham. By $H_i(X)$ we mean the dual of $H^i(X)$; it is a mixed Hodge structure with underlying rational vector space $H_i(X,\mathbb{Q})$.

The Chow group of i-dimensional algebraic cycles on X (modulo rational equivalence) is denoted by $CH_i(X)$. We write $CH_i^{\mathrm{hom}}(X)$ for the homologically trivial subgroup of $CH_i(X)$. We set $CH_i^{\mathrm{hom}}(X)_{\mathbb{Q}} := CH_i^{\mathrm{hom}}(X) \otimes \mathbb{Q}$.

All the *Ext* groups in this section are in **MHS**.

4.2. Let X be a smooth complex projective curve, and S a finite nonempty set of (complex) points of X. We identify $H^1(X)$ as a subobject of $H^1(X \setminus S)$ via the map induced by the inclusion $(X \setminus S) \subset X$. The reader can refer to Deligne's [7, §10.3] for a thorough study of the mixed Hodge structure $H^1(X \setminus S)$.

Since $X \setminus S$ is affine, every element of $H^1(X \setminus S)_{\mathbb{C}}$ can be represented by a meromorphic differential form on X with possible singularities only along S, and has a well-defined residue at every $p \in X$. Indeed, if $c = [\omega]$ with ω a meromorphic form, set $res_p(c) := res_p(\omega)$ (= the residue of ω at p, which is $1/(2\pi i)$ times the integral of ω along a small positively oriented loop around p). The subspace $H^1(X)_{\mathbb{C}}$ of $H^1(X \setminus S)_{\mathbb{C}}$ consists of the cohomology classes with zero residue everywhere (in other words, classes of differentials of the second kind).

For any vector space or mixed Hodge structure V, we denote by $(V^S)'$ the kernel of the map

$$V^S \longrightarrow V \qquad\qquad (v_p)_{p\in S} \mapsto \sum_{p\in S} v_p$$

(where the v_p are in V).

One has a short exact sequence of mixed Hodge structures

$$0 \longrightarrow H^1(X) \longrightarrow H^1(X \setminus S) \overset{2\pi i \cdot res_S}{\longrightarrow} (\mathbb{Q}(-1)^S)' \longrightarrow 0, \tag{16}$$

where the injective arrow is inclusion and $res_S : H^1(X \setminus S)_{\mathbb{C}} \to (\mathbb{C}^S)'$ is the map $c \mapsto (res_p(c))_{p\in S}$. (Recall that in our notation, the underlying rational vector space of $\mathbb{Q}(-1)$ is $\mathbb{Q}$. The factor of $2\pi i$ is included so that the map is defined over $\mathbb{Q}$.)

4.3. Recall that the cup product $H^1(X)\otimes H^1(X) \longrightarrow H^2(X)$ gives a polarization of $H^1(X)$, so that $H^1(X)$ is a semisimple object (see [9], for instance). In view of the exact sequence (16) and Section 3.5, the determination of the group $\mathcal{U}(H^1(X \setminus S))$ (= the unipotent radical of the Mumford-Tate group of $H^1(X \setminus S)$) amounts to finding

$$\underline{Lie}(\mathcal{U}(H^1(X \setminus S)))^{\perp} \subset H^1(X)^{\vee} \otimes (\mathbb{Q}(-1)^S)'.$$

We use the Poincaré duality isomorphism

$$PD : H^1(X)(1) \longrightarrow H^1(X)^\vee \qquad [\eta] \mapsto \int_X \eta \wedge - \ ,$$

where η is a closed smooth 1-form on X, and the isomorphism

$$H^1(X)(1) \otimes (\mathbb{Q}(-1)^S)' \longrightarrow (H^1(X)^S)' \qquad c \otimes (a_p)_{p\in S} \mapsto (a_p c)_{p\in S}$$

to identify

$$H^1(X)^\vee \otimes (\mathbb{Q}(-1)^S)' \cong (H^1(X)^S)'.$$

Following the notation of Section 3.5, we let μ be the element of $Ext((H^1(X)^S)', \mathbb{1})$ corresponding to the sequence (16) under the canonical isomorphism

$$Ext((\mathbb{Q}(-1)^S)', H^1(X)) \cong Ext(H^1(X)^\vee \otimes (\mathbb{Q}(-1)^S)', \mathbb{1}) = Ext((H^1(X)^S)', \mathbb{1}). \tag{17}$$

By Corollary 3.5.2 (and on recalling that $H^1(X)$ is semisimple), $\underline{Lie}(\mathcal{U}(H^1(X \setminus S)))^\perp$ is the largest subobject of $(H^1(X)^S)'$ the restriction of μ to which is trivial.

4.4. Before we proceed to study restrictions of μ, let us recall some facts about extensions of mixed Hodge structures. Let A be a pure Hodge structure of weight 1. Thanks to Carlson [6], there is a canonical isomorphism

$$Ext(A, \mathbb{1}) \longrightarrow JA^\vee = \frac{A^\vee{}_\mathbb{C}}{F^0(A^\vee{}_\mathbb{C}) + A^\vee{}_\mathbb{Q}}.$$

The image of an element of $Ext(A, \mathbb{1})$ represented by an exact sequence

$$0 \longrightarrow \mathbb{1} \longrightarrow E \longrightarrow A \longrightarrow 0$$

under this isomorphism is obtained as follows: choose a Hodge section s of the map $E_\mathbb{C} \longrightarrow A_\mathbb{C}$ (i.e. a section that is compatible with the Hodge filtration), and a retraction r of the linear map $\mathbb{C} \longrightarrow E_\mathbb{C}$ which is defined over $\mathbb{Q}$ (i.e. restricts to a map $E_\mathbb{Q} \longrightarrow \mathbb{Q}$). Then the isomorphism of Carlson sends the extension class of the sequence above to the class of $r \circ s$ in $JA^\vee$. The ambiguity of the choices of s and r is resolved by modding out by $F^0(A^\vee{}_\mathbb{C}) + A^\vee{}_\mathbb{Q}$. For the proof of this map being an isomorphism, see [6]. The isomorphism is functorial in A. (Note: In [6], Carlson proves the analogous result for integral mixed Hodge structures. The proof of the rational case is identical.)

Take $A = H^1(X)$. Then the Abel-Jacobi map gives an isomorphism

$$AJ_X : CH_0^{\hom}(X)_\mathbb{Q} \longrightarrow JH_1(X)$$

(sending the class of $p - q$, with $p, q \in X$, to the class of the functional $\int_q^p$ on the space of harmonic 1-forms on X). In view of Carlson's isomorphism, we thus have an isomorphism

$$Ext(H^1(X), \mathbb{1}) \cong CH_0^{\hom}(X)_\mathbb{Q}. \tag{18}$$

We shall identify these two groups to simplify the notation.

Back to arbitrary A of weight 1, since $A^\vee$ is pure of weight -1, the intersection of $A^\vee{}_\mathbb{R}$ and $F^0(A^\vee{}_\mathbb{C})$ is zero. Thus the inclusion $A^\vee{}_\mathbb{R} \subset A^\vee{}_\mathbb{C}$ induces an injection

$$A^\vee{}_\mathbb{R} \longrightarrow \frac{A^\vee{}_\mathbb{C}}{F^0(A^\vee{}_\mathbb{C})}.$$

Comparing dimensions as real vector spaces, we see that the above map is indeed an isomorphism. It induces an isomorphism

$$\frac{{A^\vee}_{\mathbb{R}}}{{A^\vee}_{\mathbb{Q}}} \ \longrightarrow \ JA^\vee.$$

(In particular, every element of $JA^\vee$ has a representative defined over $\mathbb{R}$, i.e. that restricts to an element of ${A^\vee}_{\mathbb{R}}$.) On the other hand, we have an isomorphism

$$\frac{{A^\vee}_{\mathbb{R}}}{{A^\vee}_{\mathbb{Q}}} \ \longrightarrow \ Hom(A_{\mathbb{Q}}, \mathbb{R}/\mathbb{Q}) \qquad\qquad [f] \ \mapsto \ f\,\Big|_{A_{\mathbb{Q}}} \pmod{\mathbb{Q}}.$$

Using the two isomorphisms above, we get an isomorphism

$$JA^\vee \ \longrightarrow \ Hom(A_{\mathbb{Q}}, \mathbb{R}/\mathbb{Q}), \tag{19}$$

which sends $[f] \in JA^\vee$ with f defined over $\mathbb{R}$ to $f\,\Big|_{A_{\mathbb{Q}}} \pmod{\mathbb{Q}}$. This isomorphism is also functorial in A.

4.5. Let $\Omega^1(X \log S)$ denote the space of differential forms of the third kind with singularities along S. The obvious map $\Omega^1(X \log S) \longrightarrow H^1(X \setminus S)$ (sending a differential form to its cohomology class) is an isomorphism onto $F^1H^1(X \setminus S)$. It is well-known that for any distinct $p, q \in X$, there exists an element $\omega_{p,q}$ of $\Omega^1(X \log\{p, q\})$ whose cohomology class in $H^1(X \setminus \{p, q\})$ is real (or equivalently, its integral along any loop in $X \setminus \{p, q\}$ is real-valued). Indeed, for instance, this can be seen from the fact that the extension (16) for $X \setminus \{p, q\}$, being an extension of two pure Hodge structures of consecutive weights, splits in the category of real mixed Hodge structures (e.g. see [14], the paragraph above Lemma (1.12)). In particular, the residues of $\omega_{p,q}$ belong to $\frac{1}{2\pi i}\mathbb{R}$, so that we can (and will) normalize $\omega_{p,q}$ such that its residues at p and q are $\frac{1}{2\pi i}$ and $-\frac{1}{2\pi i}$, respectively. Note that after the normalization, $\omega_{p,q}$ is unique (as $F^1H^1(X) \cap H^1(X)_{\mathbb{R}} = 0$).

4.6. Let us consider the restrictions of μ to some obvious subobjects of $(H^1(X)^S)'$. For each $p \in S$, let $\iota_p : H^1(X) \to H^1(X)^S$ be the embedding into the p-coordinate. Given $p, q \in S$, we have a morphism

$$\iota_p - \iota_q : H^1(X) \ \longrightarrow \ (H^1(X)^S)'$$

(which is an embedding if $p \neq q$).

Proposition 4.6.1. *Via the identification* (18)*, we have*

$$(\iota_p - \iota_q)^*(\mu) \ = \ p - q$$

(where $(\iota_p - \iota_q)^*(\mu)$ is the pullback of μ along $\iota_p - \iota_q$).

Proof. We may assume that $p \neq q$. Section 4.4 gives us isomorphisms

$$\begin{array}{ccccc} Ext(H^1(X), \mathbb{1}) & \cong & JH_1(X) & \cong & CH_0^{\hom}(X)_{\mathbb{Q}} \\ & & \wr\| & & \\ & & Hom(H^1(X)_{\mathbb{Q}}, \mathbb{R}/\mathbb{Q}). & & \end{array}$$

We shall show that the images of $(\iota_p - \iota_q)^*(\mu)$ and $p - q$ coincide in $Hom(H^1(X)_{\mathbb{R}}, \mathbb{R}/\mathbb{Q})$.

With abuse of notation, let ι_p also denote the embedding of $\mathbb{Q}(-1)$ as the p-coordinate of $\mathbb{Q}(-1)^S$. Then we have a commutative diagram

$$\begin{array}{ccc} Ext((\mathbb{Q}(-1)^S)', H^1(X)) & \xrightarrow{(\iota_p-\iota_q)^*} & Ext(\mathbb{Q}(-1), H^1(X)) \\ \wr\| & & \wr\| \\ Ext((H^1(X)^S)', \mathbb{1}) & \xrightarrow{(\iota_p-\iota_q)^*} & Ext(H^1(X), \mathbb{1}), \end{array}$$

where the vertical isomorphisms are given by (14) and Poincaré duality (the one on the left being (17)). In particular, to calculate

$$(\iota_p - \iota_q)^*(\mu) \in Ext(H^1(X), \mathbb{1}),$$

we first consider the pullback of the extension (16) along $\iota_p - \iota_q : \mathbb{Q}(-1) \longrightarrow (\mathbb{Q}(-1)^S)'$, which is the extension

(20)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(X) & \longrightarrow & (2\pi i \cdot res_S)^{-1}((\iota_p - \iota_q)(\mathbb{Q}(-1))) & \overset{2\pi i \cdot res_p}{\longrightarrow} & \mathbb{Q}(-1) & \longrightarrow & 0. \\ & & & & \| & & & & \\ & & & & H^1(X \setminus \{p,q\}) & & & & \end{array}$$

(Note that $(2\pi i \cdot res_S)^{-1}((\iota_p - \iota_q)(\mathbb{Q}(-1)))$ consists of all the cohomology classes in $H^1(X \setminus S)$ which have zero residue at every point outside of $\{p, q\}$.) Recalling the construction of the canonical isomorphism (14) from Section 3.5, the extension $(\iota_p - \iota_q)^*(\mu)$ is then given by the sequence

(21)
$$0 \longrightarrow \mathbb{1} \xrightarrow{i'} \big(H^1(X)(1) \otimes H^1(X \setminus \{p,q\}) \big) / \ker(\cup)(1) \overset{Id_{H^1(X)(1)} \otimes 2\pi i \cdot res_p}{\longrightarrow} H^1(X) \longrightarrow 0,$$

where $\ker(\cup)$ is the kernel of the cup product map $H^1(X) \otimes H^1(X) \longrightarrow H^2(X)$ (so that $\ker(\cup)(1)$ is the kernel of the evaluation pairing between $H^1(X)$ and its dual, the latter being identified with $H^1(X)(1)$), and i' is the composition

$$\mathbb{1} \xrightarrow{\simeq,\ ev} \big(H^1(X)(1) \otimes H^1(X) \big) / \ker(\cup)(1) \subset \big(H^1(X)(1) \otimes H^1(X \setminus \{p,q\}) \big) / \ker(\cup)(1).$$

Let $s : \mathbb{C} \longrightarrow H^1(X \setminus \{p,q\})_{\mathbb{C}}$ be the section of the $\mathbb{C}$-linear map $2\pi i \cdot res_p$ in the sequence (20) defined by $s(1) = [\omega_{p,q}]$ (with $\omega_{p,q}$ as in Section 4.5); it is a Hodge section defined over $\mathbb{R}$. Then

$$s' : H^1(X)_{\mathbb{C}} \longrightarrow (H^1(X)_{\mathbb{C}} \otimes H^1(X \setminus \{p,q\})_{\mathbb{C}} \big) / \ker(\cup)_{\mathbb{C}}$$

sending

$$c \mapsto c \otimes [\omega_{p,q}] \pmod{\ker(\cup)_{\mathbb{C}}}$$

is a Hodge section of $Id_{H^1(X)(1)} \otimes 2\pi i \cdot res_p$ which is defined over $\mathbb{R}$. Fix a particular lifting

(22)
$$H_1(X, \mathbb{C}) \longrightarrow H_1(X \setminus \{p,q\}, \mathbb{C}) \qquad \gamma \mapsto \widetilde{\gamma}$$

defined over $\mathbb{Q}$, and let $r : H^1(X \setminus \{p,q\})_{\mathbb{C}} \longrightarrow H^1(X)_{\mathbb{C}}$ be the dual map. Then r is a retraction of the inclusion $H^1(X)_{\mathbb{C}} \longrightarrow H^1(X \setminus \{p,q\})_{\mathbb{C}}$ defined over $\mathbb{Q}$. The map

$$r' : \big(H^1(X)_{\mathbb{C}} \otimes H^1(X \setminus \{p,q\})_{\mathbb{C}} \big) / \ker(\cup)_{\mathbb{C}} \longrightarrow \mathbb{C}$$

defined by

$$c \otimes d \pmod{\ker(\cup)} \mapsto PD(c)(r(d))$$

is a retraction of the injection i' of (21) defined over $\mathbb{Q}$. Thus $r' \circ s'$ represents the image of $(\iota_p - \iota_q)^*(\mu)$ in $JH_1(X)$. Since $r' \circ s'$ is defined over $\mathbb{R}$, the image of $(\iota_p - \iota_q)^*(\mu)$ in $Hom(H^1(X)_{\mathbb{Q}}, \mathbb{R}/\mathbb{Q})$ is the map $r' \circ s' \big|_{H^1(X)_{\mathbb{Q}}} \pmod{\mathbb{Q}}$; given a (real) harmonic form η on X with rational periods, it sends

$$[\eta] \ \mapsto \ PD([\eta])(r([\omega_{p,q}])) \pmod{\mathbb{Q}}.$$

Note that, by definition of r,

$$PD([\eta])(r([\omega_{p,q}])) \ = \int\limits_{\widetilde{\gamma_\eta}} \omega_{p,q},$$

where $\gamma_\eta \in H_1(X, \mathbb{Q})$ is the homology class such that

$$PD([\eta]) \ = \int\limits_{\gamma_\eta}$$

on $H^1(X)$ (and $\widetilde{\gamma_\eta}$ is as in (22)).

On the other hand, the image of $p - q$ in $Hom(H^1(X)_{\mathbb{Q}}, \mathbb{R}/\mathbb{Q})$ is the map

$$[\eta] \ \mapsto \int\limits_q^p \eta \pmod{\mathbb{Q}},$$

where η is a harmonic form on X with rational periods and the integral is over any path in X from q to p. Thus Proposition 4.6.1 is equivalent to the following statement: given any harmonic form η on X with rational periods,

$$\int\limits_{\widetilde{\gamma_\eta}} \omega_{p,q} \ \equiv \int\limits_q^p \eta \pmod{\mathbb{Q}}.$$

This is Lemma 9.1.2 of [13]. □

4.7. We now calculate the slightly more complicated restrictions of μ. Let

$$E \ = \ End^0(Jac(X)) \ := \ End(Jac(X)) \otimes \mathbb{Q}$$

be the endomorphism algebra of the Jacobian of X. We have an (anti-) isomorphism

$$E \ \longrightarrow \ End(H^1(X)) \qquad f \mapsto f^\star,$$

where for any element f of the endomorphism algebra of $Jac(X)$, by $f^\star$ we mean the pullback map on cohomology.[1] This induces an isomorphism

$$(E^S)' \ \longrightarrow \ Hom(H^1(X), (H^1(X)^S)') \qquad (f_p)_{p \in S} \ \mapsto \ \sum_{p \in S} \iota_p f_p^\star. \tag{23}$$

Consider the composition

$$Hom(H^1(X), (H^1(X)^S)') \ \overset{\phi \mapsto \phi^* \mu}{\longrightarrow} \ Ext(H^1(X), \mathbb{1}) \ \overset{(18)}{\cong} \ CH_0^{\text{hom}}(X)_{\mathbb{Q}} \ = \ Jac(X)(\mathbb{C}) \otimes \mathbb{Q}\,, \tag{24}$$

where we have identified $Jac(X)(\mathbb{C}) = Pic^0(X)$ with $CH_0^{\text{hom}}(X)$.

The following is a corollary of Proposition 4.6.1.

[1]We use the symbol $*$ for pullback of extensions and the symbol $\star$ for pullback of cohomology induced by morphisms of varieties.

Corollary 4.7.1. *Let* $(f_p)_{p\in S} \in (E^S)'$*. Then*

$$(\sum_{p\in S} \iota_p f_p^\star)^* \mu \;=\; \sum_{p\in S} f_p\,(p-e),$$

where e *is any point in* $X(\mathbb{C})$*.*

Proof. Let $(f_p)_{p\in S} \in (E^S)'$ and $e \in X(\mathbb{C})$. Since $\sum_{p\in S} f_p = 0$, we have

$$\sum_{p\in S} \iota_p f_p^\star \;=\; \sum_{p\in S} (\iota_p - \iota_e) f_p^\star .$$

Thus

$$\begin{aligned}(\sum_{p\in S} \iota_p f_p^\star)^* \mu \;&=\; \sum_{p\in S} ((\iota_p - \iota_e) f_p^\star)^* \mu \\ &=\; \sum_{p\in S} (f_p^\star)^* (\iota_p - \iota_e)^* \mu \\ &=\; \sum_{p\in S} f_p\,(p-e)\,,\end{aligned}$$

where in the last line we used Proposition 4.6.1 together with the commutativity of the diagram

$$\begin{array}{ccccccc} Ext(H^1(X),\mathbb{1}) & \cong & JH_1(X) & \xleftarrow{\simeq\,,\,AJ_X} & CH_0^{\mathrm{hom}}(X)_{\mathbb{Q}} & = & Jac(X)(\mathbb{C})\otimes\mathbb{Q} \\ \Big\downarrow{\scriptstyle (f^\star)^*} & & & & & & \Big\downarrow{\scriptstyle f} \\ Ext(H^1(X),\mathbb{1}) & \cong & JH_1(X) & \xleftarrow{\simeq\,,\,AJ_X} & CH_0^{\mathrm{hom}}(X)_{\mathbb{Q}} & = & Jac(X)(\mathbb{C})\otimes\mathbb{Q}\,. \end{array}$$

□

For any $(f_p)_{p\in S} \in (E^S)'$, the value of $\sum_{p\in S} f_p\,(p-e)$ does not depend on the choice of $e \in X(\mathbb{C})$. To simplify the notation, let us denote this common value by

$$\sum_{p\in S} f_p\,(p)\,.$$

Note that if

$$(f_p)_{p\in S} \;\in\; (\mathbb{Q}^S)' \;\subset\; (E^S)',$$

then $\sum_{p\in S} f_p(p)$ defined above agrees with the other possible interpretation of the notation (i.e. the image of the divisor $\sum_{p\in S} f_p p$ of degree zero with coefficients in $\mathbb{Q}$ in $CH_0^{\mathrm{hom}}(X)_{\mathbb{Q}}$).

Combining the previous corollary with Corollary 3.5.2, we immediately get:

Proposition 4.7.2. *Let* $(f_p)_{p\in S} \in (E^S)'$*. The following statements are equivalent:*

(i) *The restriction of* μ *to the image of* $\sum_{p\in S} \iota_p f_p^\star$ *splits.*

(ii) $\sum_{p\in S} f_p(p)$ *is zero in* $CH_0^{hom}(X)_{\mathbb{Q}}$*.*

(iii) *The image of* $\sum_{p\in S} \iota_p f_p^\star$ *is contained in* $\underline{Lie}(\mathcal{U}(H^1(X\setminus S)))^\perp$.

In particular, we recover the following well-known result, originally due to Deligne (see the remark below), which gives an arithmetic criterion for when $\mathcal{U}(H^1(X\setminus S))$ is trivial (or equivalently, for when the sequence (16) splits):

Corollary 4.7.3. *The group* $\mathcal{U}(H^1(X\setminus S))$ *is trivial if and only if the subgroup of the Jacobian of* X *supported on* S *has zero rank.*

Proof. Note that $\underline{Lie}(\mathcal{U}(H^1(X\setminus S)))^\perp = (H^1(X)^S)'$ if and only if $Im(\iota_p-\iota_q)$ is contained in $\underline{Lie}(\mathcal{U}(H^1(X\setminus S)))^\perp$ for every $p,q\in S$, which in turn is equivalent to $p-q$ being zero in $CH_0^{\mathrm{hom}}(X)_{\mathbb{Q}}$ for every $p,q\in S$. □

Remark 3. Corollary 4.7.3 is originally due to Deligne, implicit in [7] and announced explicitly in [8, Remarque 7.5], in relation to a new proof of the Manin-Drinfeld theorem on modular curves. See [11] for a more detailed discussion of this.

4.8. We are ready to give the main result of this part of the paper:

Theorem 4.8.1. *Let* A *be the subobject of* $(H^1(X)^S)'$ *which is the sum of the images of all the maps of the form*

$$\sum_{p\in S}\iota_p f_p^\star \ \in\ Hom(H^1(X),(H^1(X)^S)'),$$

with $(f_p)_{p\in S}\in(E^S)'$ *and* $\sum_{p\in S} f_p(p)=0$ *(see Section 4.7). Then*

$$A=\underline{Lie}(\mathcal{U}(H^1(X\setminus S)))^\perp\,.$$

Proof. The inclusion $A\subset \underline{Lie}(\mathcal{U}(H^1(X\setminus S)))^\perp$ is immediate from Proposition 4.7.2. To see the reverse inclusion, first note that since $\langle H^1(X)\rangle$ is semisimple, $\underline{Lie}(\mathcal{U}(H^1(X\setminus S)))^\perp$ is a direct sum of simple subobjects. Let B be a simple subobject of $\underline{Lie}(\mathcal{U}(H^1(X\setminus S)))^\perp$. Then B is the image of a map $H^1(X)\longrightarrow (H^1(X)^S)'$. Any such map is of the form $\sum_{p\in S}\iota_p f_p^\star$ for some $(f_p)_{p\in S}\in(E^S)'$. By Proposition 4.7.2, for the image of such a map to be in $\underline{Lie}(\mathcal{U}(H^1(X\setminus S)))^\perp$ we must have $\sum_{p\in S} f_p(p)=0$ in $CH_0^{\mathrm{hom}}(X)_{\mathbb{Q}}$. Thus $B\subset A$. □

We end the paper by deducing a result about the dimension of $\mathcal{U}(H^1(X\setminus S))$:

Theorem 4.8.2. *Let* g *be the genus of* X.

(a) *Let* D *be any division algebra contained in* E. *Then the dimension of* $\mathcal{U}(H^1(X\setminus S))$ *is at most* $2g$ *times the* D*-rank of the* D*-submodule of* $Jac(X)(\mathbb{C})\otimes\mathbb{Q}$ *generated by the subgroup supported on* S.
(b) *Suppose* $H^1(X)$ *is simple. Then the dimension of* $\mathcal{U}(H^1(X\setminus S))$ *is equal to* $2g$ *times the* E*-rank of the* E*-submodule of* $Jac(X)(\mathbb{C})\otimes\mathbb{Q}$ *generated by the subgroup supported on* S.

Proof. Let A be as in Theorem 4.8.1.

(a) For any subalgebra R of E, let Λ_R be the composition

$$(R^S)\ \hookrightarrow\ (E^S)'\ \xrightarrow{(23)}\ Hom(H^1(X),(H^1(X)^S)')\ \xrightarrow{(24)}\ Jac(X)(\mathbb{C})\otimes\mathbb{Q}\,.$$

This is R-linear by Corollary 4.7.1. The image of Λ_R is the R-submodule of $Jac(X)(\mathbb{C})\otimes\mathbb{Q}$ generated by the subgroup supported on S. Let A_R be the subobject of $(H^1(X)^S)'$

which is the sum of the images of the maps $\sum_{p\in S} \iota_p f_p^\star$ with $(f_p)_{p\in S}$ in $\ker(\Lambda_R)$, so that $A_R \subset A$ and $A_E = A$. If $\beta = \{(f_p^{(r)})_{p\in S}\}_{1\le r\le d}$ is an R-spanning set for $\ker(\Lambda_R)$, then A_R is the sum of the images of $\sum_{p\in S} \iota_p (f_p^{(r)})^\star$ for $1 \le r \le d$. Moreover, if $R = D$ is a division algebra and β is D-linearly independent, then A_D is the direct sum of the images of the previous d maps. Since each of these images is then a copy of $H^1(X)$ (because D is a division algebra), we have

$$\begin{aligned} \dim \underline{Lie}(\mathcal{U}(H^1(X\setminus S)))^\perp \;=\; \dim(A) \;&\ge\; \dim(A_D) \\ &=\; 2g\cdot \dim_D(\ker(\Lambda_D)) \\ &=\; 2g\,\left(|S|-1-\dim_D Im(\Lambda_D)\right). \end{aligned}$$

Taking orthogonal complements we get the desired bound.

(b) Since $H^1(X)$ is simple, E is a division algebra. Taking $D = E$, by the proof of Part (b) we have

$$\dim(A) \;=\; \dim(A_E) \;=\; 2g\,(|S|-1-\dim_E Im(\Lambda_E)).$$

The claimed formula follows.

□

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