MAT327H1Y: Introduction to Topology
Optional Challenge Problem #5

Note: The following is an optional, non-credit problem. It is not to be submitted, and you will **not** be tested on it. Please consider it only if you have the time and find it to be of interest.

Let $(X, \mathcal{T})$ be a topological space. A *vector bundle* $(E, \mathcal{T}_E, \pi)$ on $(X, \mathcal{T})$ consists of

(i) a topological space $(E, \mathcal{T}_E)$,
(ii) a continuous surjective function $\pi : E \to X$, and
(iii) for each $x \in X$, a real vector space structure on $\pi^{-1}(x)$,

satisfying the following property:

For each $x \in X$, there exist an open set $U \subseteq X$ containing x, a non-negative integer n, and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that[1]:

(a) The diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, "(\pi|_{\pi^{-1}(U)})|^{U}"'] & U \times \mathbb{R}^n \arrow[d, "\psi"] \\ & U \end{tikzcd}$$

commutes (ie. $\psi \circ \varphi = \pi|_{\pi^{-1}(U)}|^{U}$), where $\psi : U \times \mathbb{R}^n \to U$ is defined by $\psi(y, v) = y$.

(b) By (a), we have $\varphi(\pi^{-1}(y)) = \psi^{-1}(y)$ for all $y \in U$. We may therefore restrict the domain and codomain of φ to obtain

$$\varphi_y := (\varphi|_{\pi^{-1}(y)})|^{\psi^{-1}(y)} : \pi^{-1}(y) \to \psi^{-1}(y).$$

Note that $\pi^{-1}(y)$ is a vector space by hypothesis, and that we can regard $\psi^{-1}(y) = \{(y, v) : v \in \mathbb{R}^n\}$ as the vector space $\mathbb{R}^n$ by ignoring the first component of y. Accordingly, Condition (b) is the requirement that $\varphi_y : \pi^{-1}(y) \to \psi^{-1}(y)$ be a vector space isomorphism for each $y \in U$.

If $(X, \mathcal{T})$ is connected and $(E, \mathcal{T}_E, \pi)$ is a vector bundle on $(X, \mathcal{T})$, prove that $\dim(\pi^{-1}(x)) = \dim(\pi^{-1}(y))$ for all $x, y \in X$.

[1] Here, $\pi^{-1}(U)$ has the subspace topology coming from E, and $U \times \mathbb{R}^n$ has the product topology resulting from the subspace topology on U and the standard topology on $\mathbb{R}^n$.