RESEARCH STATEMENT

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1. Introduction

My research interests lie in the intersection of commutative algebra, representation theory and homological algebra. In particular, my major research interest is (maximal) Cohen-Macaulay representation theory. This is an active and growing research subject with connections to physics, algebraic geometry and singularity theory.

Similar to the classical theory of group representations where a group is studied by its action on vector spaces, studying the action of rings on modules can reveal facts about rings which can not otherwise be detected. So, my main interest is to study special modules over special rings which carry useful information about the geometry and singularities. Cohen-Macaulay representation theory studies maximal Cohen-Macaulay modules over Cohen-Macaulay rings. I further focus on Gorenstein rings which form a subset of Cohen-Macaulay rings. I will briefly explain these objects and try to paint a framework of my projects before I talk about them.

Let R be a commutative Noetherian ring with finite Krull dimension. Assuming that R is also local, we call it Gorenstein if it has finite injective dimension as a module over itself. We say that R is Gorenstein if every localization of it is Gorenstein. Gorenstein rings are ubiquitous [Bas63, Hun99]. The examples include regular rings, hypersurface rings and more generally complete intersection rings. The corresponding geometric objects behave well. For instance, they are locally equidimensional and connected in codimension 2 as Gorenstein rings are Cohen-Macaulay rings [Yos90]. On top of this, they have nice duality properties. For instance, in the derived category dualizing into R is actually an equivalence. The notion of Gorenstein rings can also be extended to the noncommutative world.

On Gorenstein rings, I study a special class of modules called **maximal Cohen-Macaulay modules**. We say that a finitely generated module M over a Gorenstein ring is maximal Cohen-Macaulay if its depth is equal to the dimension of the ring R. This is equivalent to the homological property that $\mathrm{Ext}^i_R(M, R) = 0$ for every $i > 0$. It turns out that this is also equivalent to being a syzygy module of arbitrarily high order. Every finitely generated projective module is maximal Cohen-Macaulay. However, the converse is true if and only if R is regular. In this sense, the category of maximal Cohen-Macaulay modules $\mathrm{MCM}(R)$ measures how singular the corresponding space is.

For any M, N in $\mathrm{MCM}(R)$, we let $P(M, N)$ to be the submodule of R-linear maps from M to N which factor through a projective module and we define $\underline{\mathrm{Hom}}_R(M, N)$ to be the quotient module $\mathrm{Hom}_R(M, N)/P(M, N)$. The quotient category $\underline{\mathrm{MCM}}(R)$ has the same objects as $\mathrm{MCM}(R)$ and the Hom-set between two modules M and N is given by $\underline{\mathrm{Hom}}_R(M, N)$. It is usually called the stable category of maximal Cohen-Macaulay modules. In this category, every projective module is identified with the zero object. Hence, $\underline{\mathrm{MCM}}(R)$ is trivial if and only if R is regular. $\underline{\mathrm{MCM}}(R)$ has a triangulated category structure given by the syzygy functor.

In [Orl09] Orlov defines the graded triangulated category of singularities of the cone over a projective variety and connects it to the bounded derived category of coherent sheaves on the base of the cone. In the Calabi-Yau case, this connection is an equivalence of categories. Moreover, the singularity category is equivalent to the stable category of graded maximal Cohen-Macaulay modules over the coordinate ring of the cone. The ungraded counterpart of this equivalence is due to Buchweitz [Buc]. **In the language of physics**, this result says that the category of graded D-branes of type B in Landau-Ginzburg models with homogeneous superpotential W is equivalent to the stable category of graded maximal Cohen-Macaulay modules over the hypersurface defined by W.

Resolution of singularities is a fundamental concept in classical algebraic geometry. A resolution of singularities replaces a singular algebraic variety by a smooth one that is isomorphic on a dense open set. In terms of algebraic geometry, this is a very useful tool. It allows the reduction of constructions and calculations to the case of a smooth variety. However, from the point of view of commutative algebra, this can seem like the end of the story. Indeed, a resolution of singularities of an affine scheme is almost never an affine scheme. So, this process replaces a well understood object with a more mysterious one from this point of view. There are several ways one can attempt to fix this problem and one of them is to allow noncommutative rings to enter the scene. However, in general the techniques of commutative algebra do not work in the noncommutative world. For instance, localization is a problem. However, there is a very nice class of rings where one might hope for a satisfactory homological theory.

Suppose R is a Gorenstein local normal domain. Then, a finitely generated R-algebra Λ is called a **noncommutative crepant resolution** if it has finite global dimension, is maximal Cohen-Macaulay as an R-module and is isomorphic to the endomorphism ring of a reflexive R-module. For a survey on noncommutative crepant resolutions, see [Leu12]. This is where Cohen-Macaulay representation theory enters the scene in noncommutative algebraic geometry.

2. On the Annihilation of Cohomology

The main theme of my thesis is a study of cohomology annihilators over Gorenstein rings. A preprint version can be found at [Ese18]. A ring element in a commutative Noetherian ring is called a cohomology annihilator if it annihilates all nth extension groups between any two finitely generated modules where n is sufficiently large.

Cohomology annihilators have been studied by several mathematicians in the last two decades [Buc, Wan94] and a theory of the cohomology annihilator ideal has been developed by Iyengar and Takahashi in a series of papers with applications to generation of triangulated categories [IT16b, IT14, IT16a]. In particular, among other significant results, Iyengar and Takahashi find bounds on the Rouquier dimension of derived (and singularity) categories of a large class of commutative rings.

Lemma 2.1. [Ese18, Lemma 2.3.] *Over Gorenstein rings, the cohomology annihilator ideal can be seen as the annihilator of the stable category of maximal Cohen-Macaulay modules.*

Hence, I study the annihilation of cohomology over commutative Gorenstein rings via maximal Cohen-Macaulay modules. In this setting, a ring element is a cohomology annihilator if and only if it annihilates the stable endomorphism ring of every finitely generated (indecomposable) maximal Cohen-Macaulay module. We will call the annihilator of the stable endomorphism ring of a module the stable annihilator of that module. Note that if the ring has finite global dimension (i.e. if the corresponding geometric object has no singularities) this ideal is equal to the ring. Hence, the cohomology annihilator ideal is an important invariant in singularity theory.

Cohomology Annihilators in Dimension One. The first main result of [Ese18] is to describe the cohomology annihilator of one dimensional Gorenstein rings.

Theorem 2.2. [Ese18, Theorem 4.4]*For a one dimensional reduced complete Gorenstein local ring, the cohomology annihilator ideal coincides with the conductor ideal of the ring.*

For a large class of commutative rings, the cohomology annihilator ideal $\mathrm{ca}(R)$ contains the Jacobian ideal $\mathrm{Jac}(R)$ of the ring [Wan94, Buc, IT16a]. The starting point for this project was an observation due to Ragnar-Olaf Buchweitz that there is a relation between vector space dimensions of $R/\mathrm{ca}(R)$ and $R/\mathrm{Jac}(R)$. Theorem 2.2 explains this observation via Milnor-Jung formula for algebraic plane curves. Building on top of Theorem 2.2, I am able to generalize this formula to double branched covers of these curves.

Theorem 2.3. [Ese18, Theorem 5.6.]*Let $R = k[\![x, y]\!]/(f)$ be the coordinate ring of a reduced curve singularity. For $S = k[\![x, y, z_1, \dots, z_l]\!]/(f + z_1^2 + \dots + z_l^2)$, we have*

$$\dim_k \frac{S}{\mathrm{Jac}(S)} = 2 \dim_k \frac{S}{\mathrm{ca}(S)} - r + 1$$

where r is the number of branches of the curve f at its singular point.

I also have found examples of hypersurfaces where the above formula does not hold.

Question 2.4. Are there interesting families of hypersurfaces where the above-mentioned formula holds true? In particular, can we compare the codimension of the cohomology annihilator ideal with the codimension of the Jacobian ideal?

Computations suggest that if the ring under consideration is the coordinate ring of the cone over a smooth elliptic curve embedded in the projective plane, then the codimension of the Jacobian ideal is twice that of the cohomology annihilator ideal. This is one direction in which I am currently working.

Cohomology Annihilators and Noncommutative Resolutions. The key lemma to achieve Theorem 2.2 is to observe that the conductor ideal coincides with the stable annihilator of the normalization of the ring. The normalization, in this setting, is finitely generated, maximal Cohen-Macaulay and is of finite global dimension.

Question 2.5. Let R be a Gorenstein ring and Λ be a finitely generated R-algebra of finite global dimension. Suppose also that Λ is maximal Cohen-Macaulay as an R-module. We know that the stable annihilator of Λ contains the cohomology annihilator ideal by Lemma 2.1. Is the converse true?

The second main result of [Ese18] is that the converse is at least geometrically true.

Theorem 2.6. [Ese18, Theorem 6.5.] *Let R be a Gorenstein ring and Λ be an R-algebra of finite global dimension δ. Suppose that Λ contains R as a direct summand. Then, $(\mathrm{ann}_R \underline{\mathrm{End}}_R(\Lambda))^{\delta+1}$ is contained in the cohomology annihilator ideal.*

That is, the cohomology annihilator ideal of R and the stable annihilator of Λ have the same radical. In every example that we have computed so far, these two ideals in fact are equal.

If Λ is, moreover, the endomorphism ring of a maximal Cohen-Macaulay module M; then the square of the stable annihilator of M is contained in the stable annihilator of Λ. Again, in all examples we computed, there is an equality between the stable annihilator of M and the cohomology annihilator ideal of R.

Motivated from cluster theory and tilting theory, cluster tilting objects in exact or triangulated categories are ubiquitous [Iya18]. For us, the importance of a d-cluster tilting maximal Cohen-Macaulay is that its endomorphism ring -called the d-Auslander algebra- gives a noncommutative crepant resolution.

Question 2.7. Let M be a d-cluster tilting object in the category of maximal Cohen-Macaulay R-modules. We know that its stable annihilator - i.e. the annihilator of its stable d-Auslander algebra - contains the cohomology annihilator ideal from Lemma 2.1. Is the converse true?

Hochschild Cohomology and Annihilation of the Singularity Category. For any associative algebra A, one can associate the Hochschild homology algebra $\mathrm{HH}_*(A, A)$, the Hochschild cohomology algebra $\mathrm{HH}^*(A, A)$ and the *stable* Hochschild cohomology algebra $\underline{\mathrm{HH}}^*(A, A)$. They are derived invariants of the algebra A, in the sense that if two algebras have equivalent derived categories, then the above constructions are isomorphic.

Let R be a commutative Gorenstein algebra of Krull dimension d. Then one can construct a long exact sequence

$$\ldots \to \mathrm{HH}_{d-i}(R, R) \to \mathrm{HH}^i(R, R) \to \underline{\mathrm{HH}}^i(R, R) \to \ldots \quad .$$

Given any M in $\mathrm{MCM}(R)$, its stable endomorphism ring is a module over $\underline{\mathrm{HH}}^0(R, R)$ in a natural way. Using the above long exact sequence, one can see that the image of the map $\mathrm{HH}_d(R, R) \to \mathrm{HH}^0(R, R) \cong R$ annihilates the stable endomorphism ring of M. As M is arbitrary, one can see that the image of this map is contained in the cohomology annihilator ideal of R.

Hence, I also think about the following type of questions:

Question 2.8. Can we compute the image of the map $\mathrm{HH}_d(R,R) \to \mathrm{HH}^0(R,R)$? If R is a hypersurface ring, then the answer is yes: the image is exactly equal to the Jacobian ideal of R, generated by the partial derivatives of the polynomial which defines R. The next step is to understand the image for complete intersections.

Question 2.9. So far, we have been interested in the action of $R \cong \mathrm{HH}^0(R,R)$ on the singularity category. However, there is also an action of the entire Hochschild cohomology algebra $\mathrm{HH}^*(A,A)$ on the singularity category. What is the image of the map $\mathrm{HH}_{d-*}(R,R) \to \mathrm{HH}^*(R,R)$? How far is it from being equal to the ideal of elements in $\mathrm{HH}^*(R,R)$ which annihilate the singularity category of R?

3. On Dominant Dimensions

Maximal Cohen-Macaulay representation theory of Cohen-Macaulay local rings and representation theory of finite dimensional algebras have a very important tool in common: the Auslander-Reiten theory. Hence, a lot of techniques can be transferred between these two theories and can also be generalized to extriangulated categories. One example is Gabriel's theorem on quiver representations and its commutative counterpart due to Buchweitz-Greuel-Schreyer.

One project that I am currently working on relies on this fact. In particular, this project tries to understand the notion of dominant dimension in maximal Cohen-Macaulay representation theory. This is an **ongoing project** joint with Graham Leushcke.

Dominant Dimension of a Finite Dimensional Algebra. Let A be a finite dimensional (or more generally an artinian) algebra. Consider A as a right module over itself and let

$$0 \to A \to I^0 \to I^1 \to \ldots \to I^m \to \ldots$$

be a minimal injective resolution of A. Then, the dominant dimension of A is defined to be the largest number k or ∞ such that $I^0, I^1, \ldots, I^{k-1}$ are projective.

Dominant dimension was introduced by Nakayama in his study of complete homology theory and has been studied intensively over the decades. For instance, it has been used to classify finite dimensional algebras of finite representation type [ARS95]. Unlike other notions of dimension, it is desired that the dominant dimension is large. This is related to self-orthogonality which plays an important role in Iyama's higher Auslander-Reiten theory and Rouquier's cover theory (See [FKY18] and references within).

Dominant Dimension Relative to Maximal Cohen-Macaulay Modules. Let R be a Cohen-Macaulay local ring with canonical module ω. Let Λ be an R-order *i.e.* a finite R-algebra which is maximal Cohen-Macaulay as an R-module. We will write $X \in \mathrm{MCM}(\Lambda)$ to mean that the left Λ-module X is maximal Cohen-Macaulay as an R-module.

We say that a finitely generated left Λ-module X is relatively injective if

1. $X \in \mathrm{MCM}(\Lambda)$ and

2. $\operatorname{Ext}^n_\Lambda(-, X) = 0$ on $\mathrm{MCM}(\Lambda)$ for any $n > 0$.

Lemma 3.1. *Let $\omega_\Lambda = \operatorname{Hom}_R(\Lambda^{\mathrm{op}}, \omega)$. Then a left Λ-module is relatively injective if and only if it is isomorphic to a direct summand of finitely many copies of ω_Λ.*

If Λ is a non-singular order in terms of Iyama and Wemyss, then ω_Λ is a projective left Λ-module [IW14]. In this case, a left Λ-module is relatively injective if and only if it is projective which is equivalent to being maximal Cohen-Macaulay.

A relatively injective resolution of Λ is a complex $0 \to I^0 \to I^1 \to \ldots \to I^m \to \ldots$ whose homology is isomorphic to Λ concentrated in degree zero where $I_0, \ldots, I^m$ are relatively injective left Λ-modules. The relative injective dimension of Λ is the infimum of lengths of relatively injective resolutions of Λ. Note that this is equal to the projective dimension of $\operatorname{Hom}_R(\Lambda, \omega)$ as a left Λ^{op}-module.

Let $0 \to I^0 \to I^1 \to \ldots \to I^{m-1} \to \ldots$ be a minimal relatively injective resolution of Λ. We say that Λ has relative dominant dimension at least m if $I^0, \ldots, I^{m-1}$ are also projective. We say that the relative dominant dimension is equal to m if it is at least m but it is not greater than m.

Relative Injective Dimension and Global Dimension. We can relate the relative injective dimension of Λ to its global dimension.

Lemma 3.2. *Let R be an R-order. Then, the relative injective dimension of Λ is $\operatorname{gldim} \Lambda - \dim R$.*

This yields the following questions.

Question 3.3. Can we construct R-orders with relative dominant dimension m and global dimension $m + \dim R$? Can we classify such orders?

Relative Dominant Dimension and Tilting. Our motivation to begin this project is tilting theory results similar to [CBS17, PS18, NRTZ17] We have some preliminary results and some approachable questions.

We say that a Λ-module X is tilting if

1. it has finite projective dimension,
2. $\operatorname{Ext}^i(X, X) = 0$ for $i > 0$, and
3. there is an exact sequence $0 \to \Lambda \to Y_0 \to Y_1 \ldots \to Y_l \to 0$ where each Y_i is a direct summand of finitely many copies of X.

Suppose that Λ is an R-order with relative dominant dimension at least m and let

$$0 \to \Lambda \to I^0 \to I^1 \to \ldots \to I^{m-1}$$

be an exact sequence with projective-relatively injective $I^0, \ldots, I^{m-1}$.

Lemma 3.4. *1. If $m > \dim R$, then the image K_j of the map $I^j \to I^{j+1}$ is maximal Cohen-Macaulay for any $j = 0, \ldots, m - \dim R - 1$. Hence, $T_j = I^0 \oplus I^1 \oplus \ldots \oplus I^j \oplus K_j$ is a tilting module.*

Question 3.5. Can we prove any uniqueness results for these tilting modules? For instance, can we prove that T_j is the unique tilting module (up to isomorphism) which has projective dimension $j+1$?

Analogous results in the theory of finite dimensional algebras exist [CBS17, NRTZ17].Unfortunately, we do not have some of the machinery which goes into the proofs of those results.

Question 3.6. Can we prove that the tilting module T_j is also cotilting?

We have finite injective dimension and self-orthogonality, and in this case T_j is called a special cotilting module in the literature.

Question 3.7. The importance of tilting modules comes from the fact that they produce derived equivalences. In particular, we know that Λ is derived equivalent to $\mathrm{End}_\Lambda(T_j)$. What does this endomorphism ring look like?

This question, stated last, is the starting point of this project. Indeed, this ongoing project started as an attempt to understand the projective quotient algebra which appears in [CBS17, PS18] in the finite dimensional algebra case. The projective quotient algebra is defined as a certain endomorphism algebra in the homotopy category of two term complexes of the form $P \to X$ where P is a projective module and X is a finite dimensional module. It is then showed to be a tilted algebra and used as a desingularization of quiver varieties. The corresponding tilting module is analogous to our T_j and we believe that the answer to Question 3.8 will be analogous to the projective quotient algebra.

Finite CM Type. We know that if Λ is a non-singular order, then the canonical module ω_Λ of Λ (and every other maximal Cohen-Macaulay Λ-module) is projective [IW14]. We are interested in the case where Λ has finite global dimension but fails to be non-singular.

Example 3.8. Let k be an infinite field and let R be the quotient of $k[\![x, y, z, u, v]\!]$ by the ideal $(xz - y^2, xv - yu, yv - zu)$. Then, R is a 3-dimensional isolated singularity and it is not-Gorenstein. Let $\Lambda = \mathrm{End}_R(R \oplus \omega)$. Then Λ is an R-order of global dimension 4. Λ is not a non-singular order.

Question 3.9. Can we classify R-orders which have finitely many non-projective maximal Cohen-Macaulay modules up to isomorphism - using the notion of relative dominant dimension?

The analogous result in the finite dimensional algebra case is proved in [ARS95]. Even though some details are missing in our case, we can answer this question positively.

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