Jordan Form

(1) Find the Jordan for the following operators.

(a) \( T : V \to V \) defined by \( T(f) = f' \) where \( V \) is the vectorspace of real valued functions spanned by \( \{1, t, t^2, e^t, te^t\} \).

(b) \( T : M_2(\mathbb{R}) \to M_2(\mathbb{R}) \) defined by

\[
T(A) = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} A - A^t
\]

(c) Let \( P_2[x, y] \) be the space of polynomials with coefficients from a field \( F \) of degree at most 2 (so, it is a 6-dimensional space with a basis \( \{1, x, y, x^2, xy, y^2\} \)). Let \( T : P_2[x, y] \to P_2[x, y] \) be defined by

\[
T(p) = \frac{\partial}{\partial x} p + \frac{\partial}{\partial y} p
\]

(2) Let \( T \) be a linear operator on a finite dimensional vectorspace over a perfect field (just consider \( \mathbb{C} \)). Prove that \( T \) can be written as a sum of a semisimple (you can think this as diagonalizable over \( \mathbb{C} \)) operator \( S \) and a nilpotent operator \( N \). Moreover, prove that \( SN - NS = 0 \).

(3) Let \( A \) be a \( 3 \times 3 \) matrix. Discuss how one can compute \( A^{100} \) with as little computation as possible.

(4) Classify all matrices in \( M_n(\mathbb{C}) \) up to similarity where \( n = 1, 2, 3, 4, 5 \).