More on Spectral Theorem, Introduction to Jordan Canonical Forms

(1) Let V be a finite dimensional complex inner product space. Give an example of a nonzero nilpotent normal linear operator T on V or prove that such an operator doesn't exist.

(2) Let V be a finite dimensional complex inner product space. A linear operator T on V is called positive-definite if $\langle T(x), x\rangle > 0$ for all $x \neq 0$ and positive semi-definite if $\langle T(x), x\rangle \geq 0$ for all x.

- **(a)** Show that for any T, T^*T is positive semi-definite. When is positive-definite?
- **(b)** Show that if T is an orthogonal projection onto a subspace W, then T is positive semi-definite. For which W, is it positive-definite?

(3) Let T be a normal operator on a finite dimensional complex inner product space. By Spectral Theorem, we know that T can be written as $T = \lambda_1\pi_1 + \ldots + \lambda_k\pi_k$ where λ_js are distinct eigenvalues and π_j is the orthogonal projection onto the eigenspace corresponding to λ_j.

- **(a)** Show that for any $j = 1, \ldots, k$ and for any natural number $n > 0$, we have $\pi_j^n = \pi_j$.
- **(b)** Show that for any $i \neq j$, we have $(\pi_i + \pi_j)^2 = \pi_i + \pi_j$.
- **(c)** Show that for any $i \neq j$, we have $(\lambda_i\pi_i + \lambda_j\pi_j)^2 = \lambda_i^2\pi_i + \lambda_j^2\pi_j$
- **(d)** Show that for any polynomial g, we have $g(T) = g(\lambda_1)\pi_1 + \ldots + g(\lambda_k)\pi_k$.
- **(e)** Show that $T^* = \overline{\lambda_1}\pi_1 + \ldots + \overline{\lambda_k}\pi_k$.
- **(f)** Prove that there is a polynomial g such that $g(T) = T^*$.

(4) Label the following statements as true or false and justify your answer.

- **(a)** Let V be a finite dimensional complex inner product space and T be a linear operator on V. If T is diagonalizable, then is orthonormally diagonalizable.
- **(b)** Let V be a finite dimensional complex vectorspace. Suppose that $(V, \langle -, -\rangle_1)$ and $(V, \langle -, -\rangle_2)$ are two inner product space structures on V. If a linear operator is normal with respect to the first structure, then it is normal with respect to the other structure. (In other words, being normal is independent of the inner product we put on V).
- **(c)** Let T be a linear operator on a finite-dimensional vectorspace V. There is a basis consisting of eigenvectors of T.
- **(d)** Let T be a linear operator on a finite-dimensional vectorspace V such that the characteristic polynomial of T splits. Then, there is a basis consisting of eigenvectors of T.
- **(e)** Let T be a linear operator on a finite-dimensional vectorspace V such that the characteristic polynomial of T splits. Then, there is a spanning set for V consisting of eigenvectors of T, but it is not necessarily a basis.
- **(f)** Let T be a linear operator on a finite-dimensional vectorspace V such that the characteristic polynomial of T splits. Then, there is a spanning set for V consisting of generalized eigenvectors of T.
- **(g)** Let T be a linear operator on a finite-dimensional vectorspace V and λ be an eigenvalue of T. The generalized eigenspace K_λ is a subspace of V.

(h) Let T be a linear operator on a finite-dimensional vectorspace V and λ be an eigenvalue of T. The generalized eigenspace K_λ is not necessarily a T-invariant subspace of V.

(g) Let λ_1, λ_2 be two distinct eigenvalues of a linear operator T on a finite dimensional vectorspace V. Then, the restriction of $T - \lambda_1 I$ to K_{λ_2} is one-to-one.

(h) Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of a linear operator T on a finite dimensional vectorspace V. If T is diagonalizable, then $K_{\lambda_i} \cap K_{\lambda_j} = \emptyset$ for $i \neq j$.

(i) Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of a linear operator T on a finite dimensional vectorspace V. If $K_{\lambda_i} \cap K_{\lambda_j} = \emptyset$ for $i \neq j$, then T is diagonalizable.

(5) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear operator given by $T(v) = Av$ where A is the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & -4 \\ 1 & -2 & -1 & -4 \\ -1 & 2 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Compute A^2.

(b) Conclude T is nilpotent. Can T be diagonalizable?

(c) Find a canonical basis α and compute $[T]_\alpha$.