Inner Products

(1) Label the following statements as true or false and justify your answer.

(a) Let V be an inner product space and $T : V \to V$ be a linear operator. If W is a T-invariant subspace, then it is also a T^*-invariant subspace.

(b) Let V be an inner product space and $T : V \to V$ be a linear operator. If W is a T-invariant subspace, then $W^\perp$ is also a T-invariant subspace.

(c) Let V be an inner product space and $T : V \to V$ be a linear operator. If W is a T-invariant subspace, then $W^\perp$ is a T^*-invariant subspace.

(2) (From MAT224 Winter 2014) Let $\alpha = \{(1,0,1),(1,1,1),(0,0,1)\}$ be a basis for $\mathbb{C}^3$ with its standar inner product space structure. Let $f : \mathbb{C}^3 \to \mathbb{C}^3$ be the function defined by

$$[f]_\alpha = \begin{bmatrix} 0 & i & 1 \\ -i & 1 & 1+i \\ 1 & 1-i & -1 \end{bmatrix}$$

(a) Show that $[f]_\alpha$ is a self-adjoint matrix.

(b) Show that f is not a self-adjoint transformation.

(3) (From MAT224 Winter 2014) Consider $V = P_2(\mathbb{C})$ together with inner product

$$\langle p, q\rangle = p(0)\overline{q(0)} + p'(0)\overline{q'(0)} + \frac{1}{4}p''(0)\overline{q''(0)}$$

Let $f : V \to V$ be the linear map defined by

$$f(a_0 + a_1x + a_2x^2) = (a_1 - ia_2) + (a_0 - ia_2)x + (ia_0 + ia_1)x^2$$

(a) Show that f is a self-adjoint transformation.

(b) Consider the basis $\beta = \{1, 1+x, 1+x^2\}$. Show that $[f]_\beta$ is not a self-adjoint matrix.

(c) Find an orthonormal basis for V consisting of eigenvectors of f.

(4) (From MAT224 Winter 2014) Consider $P_2(\mathbb{R})$ with inner product

$$\langle p(-1)q(-1) + p(0)q(0) + p(1)q(1)\rangle$$

and let $\partial : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the differentiation map $\partial(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x$. Find ∂^*.