INNER PRODUCTS

(1) Label the following statements as true or false and justify your answer.

(a) Let \( V \) be an inner product space and \( T : V \to V \) be a linear operator. If \( W \) is a \( T \)-invariant subspace, then it is also a \( T^* \)-invariant subspace.
(b) Let \( V \) be an inner product space and \( T : V \to V \) be a linear operator. If \( W \) is a \( T \)-invariant subspace, then \( W^\perp \) is also a \( T \)-invariant subspace.
(c) Let \( V \) be an inner product space and \( T : V \to V \) be a linear operator. If \( W \) is a \( T \)-invariant subspace, then \( W^\perp \) is a \( T^* \)-invariant subspace.

(2) (From MAT224 Winter 2014) Let \( \alpha = \{ (1,0,1), (1,1,1), (0,0,1) \} \) be a basis for \( \mathbb{C}^3 \) with its standard inner product space structure. Let \( f : \mathbb{C}^3 \to \mathbb{C}^3 \) be the function defined by
\[
[f]_\alpha = \begin{bmatrix}
0 & i & 1 \\
-i & 1 & 1+i \\
1 & 1-i & -1
\end{bmatrix}
\]

(a) Show that \([f]_\alpha\) is a self-adjoint matrix.
(b) Show that \( f \) is not a self-adjoint transformation.

(3) (From MAT224 Winter 2014) Consider \( V = P_2(\mathbb{C}) \) together with inner product
\[
\langle p, q \rangle = p(0)\overline{q(0)} + p'(0)\overline{q'(0)} + \frac{1}{4}p''(0)\overline{q''(0)}
\]
Let \( f : V \to V \) be the linear map defined by
\[
f(a_0 + a_1x + a_2x^2) = (a_1 - ia_2) + (a_0 - ia_2)x + (ia_0 + ia_1)x^2
\]

(a) Show that \([f]_\beta\) is a self-adjoint matrix.
(b) Show that \( f \) is not a self-adjoint transformation.
(c) Consider the basis \( \beta = \{ 1, 1+x, 1+x^2 \} \). Show that \([f]_\beta\) is not a self-adjoint matrix.
(c) Find an orthonormal basis for \( V \) consisting of eigenvectors of \( f \).

(4) (From MAT224 Winter 2014) Consider \( P_2(\mathbb{R}) \) with inner product
\[
\langle p(-1)q(-1) + p(0)q(0) + p(1)q(1) \rangle
\]
and let \( \partial : P_2(\mathbb{R}) \to P_2(\mathbb{R}) \) be the differentiation map \( \partial(a_0 + a_1x + a_2x^2) = a_1 + 2a_2x \). Find \( \partial^* \).