INNER PRODUCTS

(1) Label the following statements as true or false and justify your answer.

(a) Let \( V \) be a finite dimensional real inner product space and \( T : V \to V \) be a linear operator. If \( \langle T(x), x \rangle = 0 \) for all \( x \in V \), then \( T \) is the zero transformation.

(b) Let \( V \) be a finite dimensional complex inner product space and \( T : V \to V \) be a linear operator. If \( \langle T(x), x \rangle = 0 \) for all \( x \in V \), then \( T \) is the zero transformation.

(c) Let \( V \) be a finite dimensional inner product space and \( T : V \to V \) be a linear operator. If \( \langle T(x), y \rangle = \langle x, y \rangle \) for all \( x, y \in V \), then \( T \) is the identity matrix.

(2) Let \( V \) be a finite dimensional inner product space and \( T : V \to V \) be a linear operator. Prove that if \( \langle T(x), T(y) \rangle = \langle x, y \rangle \) for all \( x, y \in V \), then \( T \) is invertible and \( T^{-1} = T^* \).

(3) Let \( V \) be a finite dimensional inner product space and \( \{v_1, \ldots, v_n\} \) be an orthonormal basis for \( V \). Prove Parseval’s identity: For any \( x, y \in V \),

\[
\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}
\]

(4) Let \( V \) be an inner product space and let \( S \) be a finite orthonormal subset of \( V \) (Without loss of generality suppose \( S \) does not contain zero). Prove Bessel’s Inequality: For any \( x \in V \),

\[
|x|^2 \geq \sum_{v \in S} |\langle x, v \rangle|^2
\]

(5) Let \( V = P_3(\mathbb{R}) \) equipped with the inner product

\[
\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)dt
\]

Find an orthonormal basis for \( W = \text{span}\{1, x^2\} \) and \( W^\perp \).