Inner Products

(1) Label the following statements as true or false and justify your answer.

(a) Let V be a finite dimensional real inner product space and $T : V \to V$ be a linear operator. If $\langle T(x), x\rangle = 0$ for all $x \in V$, then T is the zero transformation.

(b) Let V be a finite dimensional complex inner product space and $T : V \to V$ be a linear operator. If $\langle T(x), x\rangle = 0$ for all $x \in V$, then T is the zero transformation.

(c) Let V be a finite dimensional inner product space and $T : V \to V$ be a linear operator. If $\langle T(x), y\rangle = \langle x, y\rangle$ for all $x, y \in V$, then T is the identity matrix.

(2) Let V be a finite dimensional inner product space and $T : V \to V$ be a linear operator. Prove that if $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$, then T is invertible and $T^{-1} = T^*$.

(3) Let V be a finite dimensional inner product space and $\{v_1, \ldots, v_n\}$ be an orthonormal basis for V. Prove *Parseval's identity*: For any $x, y \in V$,

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}$$

(4) Let V be an inner product space and let S be a finite orthonormal subset of V (Without loss of generality suppose S does not contain zero). Prove *Bessel's Inequality*: For any $x \in V$,

$$|x|^2 \geq \sum_{v \in S} |\langle x, v\rangle|^2$$

(5) Let $V = P_3(\mathbb{R})$ equipped with the inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(t)q(t)dt$$

Find an orthonormal basis for $W = span\{1, x^2\}$ and $W^{\perp}$.