Eigenvectors, Inner Products

(1) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be an injective linear map. Show that there is a line through origin in $\mathbb{R}^3$ which stays fixed under this transformation.

(2) Consider the following matrix

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$$

- **(a)** Show that if a, b are integers, then $A(a, b)^T$ is also a vector with integer coordinates (A sends the integer lattice to itself).
- **(b)** Show that $(\sqrt{2}, 1)^T$ is an eigenvector with eigenvalue $\sqrt{2} - 1$.
- **(c)** Show that $0 < \sqrt{2} - 1 < 1$.
- **(d)** Suppose, for a contradiction, that $\sqrt{2} = \dfrac{m}{n} \in \mathbb{Q}$. Then, $n\sqrt{2} = m \in \mathbb{Z}$ and $(n\sqrt{2}, n)^T$ is an eigenvector with eigenvalue $\sqrt{2} - 1$ and it has integer coordinates. Argue that this means (i) $A^k(n\sqrt{2}, 2)^T = (\sqrt{2} - 1)^k(n\sqrt{2}, n)^T$ for all $k \geq 1$ and (ii) $A^k(n\sqrt{2}, n)^T$ also has integer coefficients. (Hint: Apply part a for (ii)).
- **(e)** Show that part (d) yields a contradiction if we choose k *very big*. Conclude that $\sqrt{2}$ is irrational. (Kalman, D. Variations on an Irrational Theme-Geometry, Dynamics, Algebra, Mathematics Magazine, Vol. 70, No. 2 (Apr., 1997), pp. 93-104)

(3) Let $T : V \to V$ be a linear operator on a complex inner product space with the property that $\|T(x)\| = \|x\|$ for all $x \in V$. What can you say about T?

(4) Let V and W be complex vectorspaces with an injective linear map $f : V \to W$ between them. Suppose that W is equipped with an inner product $\langle -, - \rangle_W$. Define a map $\langle -, - \rangle_V$ from $V \times V$ to $\mathbb{C}$ by the formula $\langle v_1, v_2 \rangle_V = \langle f(v_1), f(v_2) \rangle_W$. Show that $\langle -, - \rangle_V$ is an inner product.

(5) Let V be a finite dimensional inner product space and $v_0 \in V$ be fixed.

- **(a)** Show that the rule $\phi_0(v) = \langle v, v_0 \rangle$ defines a linear functional on V.
- **(b)** Let $\phi \in V^*$ be a linear functional on V. Is it true that there necessarily exists a $v_0 \in V$ such that $\phi(v) = \langle v, v_0 \rangle$ for all $v \in V$?

(6) Let V be an inner product space and W be a subspace with a basis $\alpha = \{w_1, \ldots, w_m\}$. Define

$$W^\perp = \{v \in V : \langle v, w \rangle = 0 \text{ for all } w \in W\}$$

- **(a)** Show that $W^\perp$ is a subspace.
- **(b)** Show that $v \in W^\perp$ if and only if $\langle v, w_i \rangle = \langle w_i, v \rangle = 0$ for $i = 1, \ldots, m$.

(7) Let Q be the set of all matrices in $M_{2\times 2}(\mathbb{C})$ with the following property:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$$

Show that Q is a subspace and find a basis for $Q^\perp$.

(8) Label the following statements as true or false and justify your answer.

(a) Let V be a finite dimensional inner product space and W be a subspace of V. If $V = W \oplus U$ for a subspace U of W, then U is the orthogonal complement of W.

(b) Let V be a vector space and W_1, W_2, W_3 be subspaces of V. If $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_3$, then we have $W_2 = W_3$.

(c) Let V be a finite dimensional vector space and U, W be subspaces of V. Let α, β be bases for U and W, respectively. If $|\alpha \cup \beta| = \dim V$, then $V = U \oplus W$.

(d) Let V be a finite dimensional vector space and U, W be subspaces of V. Let α, β be bases for U and W, respectively. If $|\alpha \cup \beta| = \dim V$ and $\alpha \cup \beta$ is linearly independent, then $V = U \oplus W$.

(e) Let V be a finite dimensional inner product space and $T : V \to V$ be a linear operator. If $\langle T(x), x \rangle = 0$ for all $x \in V$, then T is the zero transformation.

(f) Let V be a finite dimensional inner product space and $T : V \to V$ be a linear operator. If $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$, then T is invertible.

(g) Let V be a finite dimensional vector space and $T : V \to V$ be a linear operator. Then, $V = \ker T \oplus \operatorname{im} T$.

(h) Let V be a finite dimensional inner product space and $T : V \to V$ be a linear operator. Let β be a basis for V. If $[T]_\beta$ is a self-adjoint matrix, then T is a self-adjoint transformation.