Eigenvectors, Inner Products

(1) Let \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) be an injective linear map. Show that there is a line through origin in \( \mathbb{R}^3 \) which stays fixed under this transformation.

(2) Consider the following matrix
\[
A = \begin{bmatrix}
-1 & 2 \\
1 & -1
\end{bmatrix} \in M_{2 \times 2}(\mathbb{R})
\]
(a) Show that if \( a, b \) are integers, then \( A(a, b)^T \) is also a vector with integer coordinates (\( A \) sends the integer lattice to itself).
(b) Show that \((\sqrt{2}, 1)^T\) is an eigenvector with eigenvalue \( \sqrt{2} - 1 \).
(c) Show that \( 0 < \sqrt{2} - 1 < 1 \).
(d) Suppose, for a contradiction, that \( \sqrt{2} = \frac{m}{n} \in \mathbb{Q} \). Then, \( n\sqrt{2} = m \in \mathbb{Z} \) and \((n\sqrt{2}, n)^T\) is an eigenvector with eigenvalue \( \sqrt{2} - 1 \) and it has integer coordinates. Argue that this means (i) \( A^k(n\sqrt{2}, 2)^T = (\sqrt{2} - 1)^k(n\sqrt{2}, n)^T \) for all \( k \geq 1 \) and (ii) \( A^k(n\sqrt{2}, n)^T \) also has integer coefficients. (Hint: Apply part a for (ii)).
(e) Show that part (d) yields a contradiction if we choose \( k \) very big. Conclude that \( \sqrt{2} \) is irrational. (Kalman, D. Variations on an Irrational Theme-Geometry, Dynamics, Algebra, Mathematics Magazine, Vol. 70, No. 2 (Apr., 1997), pp. 93-104)

(3) Let \( T : V \to V \) be a linear operator on a complex inner product space with the property that \( \|T(x)\| = \|x\| \) for all \( x \in V \). What can you say about \( T \)?

(4) Let \( V \) and \( W \) be complex vectorspaces with an injective linear map \( f : V \to W \) between them. Suppose that \( W \) is equipped with an inner product \( \langle -, - \rangle_W \). Define a map \( \langle -, -, \rangle_V \) from \( V \times V \) to \( \mathbb{C} \) by the formula \( \langle v_1, v_2 \rangle_V = \langle f(v_1), f(v_2) \rangle_W \). Show that \( \langle -, -, \rangle_V \) is an inner product.

(5) Let \( V \) be a finite dimensional inner product space and \( v_0 \in V \) be fixed.
(a) Show that the rule \( \phi_0(v) = \langle v, v_0 \rangle \) defines a linear functional on \( V \).
(b) Let \( \phi \in V^* \) be a linear functional on \( V \). Is it true that there necessarily exists a \( v_0 \in V \) such that \( \phi(v) = \langle v, v_0 \rangle \) for all \( v \in V \)?

(6) Let \( V \) be an inner product space and \( W \) be a subspace with a basis \( \alpha = \{w_1, \ldots, w_m\} \). Define \( W^\perp = \{v \in V : \langle v, w \rangle = 0 \text{ for all } w \in W\} \)

(a) Show that \( W^\perp \) is a subspace.
(b) Show that \( v \in W^\perp \) if and only if \( \langle v, w_i \rangle = \langle w_i, v \rangle = 0 \) for \( i = 1, \ldots, m \).

(7) Let \( Q \) be the set of all matrices in \( M_{2 \times 2}(\mathbb{C}) \) with the following property:
\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} = \begin{bmatrix}
b & d \\
a & c
\end{bmatrix}
\]
Show that \( Q \) is a subspace and find a basis for \( Q^\perp \).
(8) Label the following statements as true or false and justify your answer.

(a) Let $V$ be a finite dimensional inner product space and $W$ be a subspace of $V$. If $V = W \oplus U$ for a subspace $U$ of $W$, then $U$ is the orthogonal complement of $W$.

(b) Let $V$ be a vector space and $W_1, W_2, W_3$ be subspaces of $V$. If $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_3$, then we have $W_2 = W_3$.

(c) Let $V$ be a finite dimensional vector space and $U, W$ be subspaces of $V$. Let $\alpha, \beta$ be bases for $U$ and $W$, respectively. If $|\alpha \cup \beta| = \dim V$, then $V = U \oplus W$.

(d) Let $V$ be a finite dimensional vector space and $U, W$ be subspaces of $V$. Let $\alpha, \beta$ be bases for $U$ and $W$, respectively. If $|\alpha \cup \beta| = \dim V$ and $\alpha \cup \beta$ is linearly independent, then $V = U \oplus W$.

(e) Let $V$ be a finite dimensional inner product space and $T : V \to V$ be a linear operator. If $\langle T(x), x \rangle = 0$ for all $x \in V$, then $T$ is the zero transformation.

(f) Let $V$ be a finite dimensional inner product space and $T : V \to V$ be a linear operator. If $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$, then $T$ is invertible.

(g) Let $V$ be a finite dimensional vector space and $T : V \to V$ be a linear operator. Then, $V = \ker T \oplus \operatorname{im} T$.

(h) Let $V$ be a finite dimensional inner product space and $T : V \to V$ be a linear operator. Let $\beta$ be a basis for $V$. If $[T]_\beta$ is a self-adjoint matrix, then $T$ is a self-adjoint transformation.