Eigenvectors, Eigenvalues, Eigenspaces

(1) Let T be a linear operator on a vector space V. Suppose that λ_1, λ_2 are two eigenvalues of T, with corresponding eigenvectors v_1, v_2. Show that if $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are linearly independent. The converse of this is not true, identity transformation is a counterexample. We can generalize this result. Show that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are n distinct eigenvalues of a linear operator T on an n-dimensional vector space V, then V has a basis consisting of eigenvectors of T.

(2) Suppose that V is an n-dimensional vector space. Consider a linear operator $T \in \mathrm{End}(V)$ such that it has n-linearly independent eigenvectors: $\{v_1, \ldots, v_n\}$. Since $\dim V = n$, this is a basis of V. Write the matrix of this linear transformation with respect to this basis. What do you observe?

(3) Suppose that V is an n-dimensional vector space and $T \in \mathrm{End}(V)$. Suppose that $\alpha = \{v_1, \ldots, v_n\}$ is a basis of V such that $[T]_\alpha^\alpha$ is a diagonal matrix. What can you say about the v_i's?

(4) Find $A \in M_2(\mathbb{R})$ such that

- **(a)** A is diagonalizable and A is invertible.
- **(b)** A is diagonalizable and A is not invertible.
- **(c)** A is not diagonalizable and A is invertible.
- **(d)** A is not diagonalizable and A is not invertible.

(5) Let A be a 2×2 matrix and P_A be its characteristic polynomial. Show that

$$P_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A).$$

(6) Compute A^{100} where

$$A = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix}$$