Eigenvectors, Eigenvalues, Eigenspaces

(1) Let $T$ be a linear operator on a vector space $V$. Suppose that $\lambda_1, \lambda_2$ are two eigenvalues of $T$, with corresponding eigenvectors $v_1, v_2$. Show that if $\lambda_1 \neq \lambda_2$, then $v_1$ and $v_2$ are linearly independent. The converse of this is not true, identity transformation is a counterexample. We can generalize this result. Show that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are $n$ distinct eigenvalues of a linear operator $T$ on an $n$-dimensional vector space $V$, then $V$ has a basis consisting of eigenvectors of $T$.

(2) Suppose that $V$ is an $n$-dimensional vector space. Consider a linear operator $T \in \text{End}(V)$ such that it has $n$-linearly independent eigenvectors: $\{v_1, \ldots, v_n\}$. Since $\dim V = n$, this is a basis of $V$. Write the matrix of this linear transformation with respect to this basis. What do you observe?

(3) Suppose that $V$ is an $n$-dimensional vector space and $T \in \text{End}(V)$. Suppose that $\alpha = \{v_1, \ldots, v_n\}$ is a basis of $V$ such that $[T]_{\alpha}^{\alpha}$ is a diagonal matrix. What can you say about the $v_i$’s?

(4) Find $A \in M_2(\mathbb{R})$ such that
   
   (a) $A$ is diagonalizable and $A$ is invertible.
   (b) $A$ is diagonalizable and $A$ is not invertible.
   (c) $A$ is not diagonalizable and $A$ is invertible.
   (d) $A$ is not diagonalizable and $A$ is not invertible.

(5) Let $A$ be a $2 \times 2$ matrix and $P_A$ be its characteristic polynomial. Show that

\[ P_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A). \]

(6) Compute $A^{100}$ where

\[ A = \begin{bmatrix} 4 & -5 \\ 3 & -4 \end{bmatrix} \]