The Minimal Polynomial

(1) Let F be an algebraically closed field and consider the conjugation action of $GL(n, F)$ on $M_n(F)$. Show that if p is a polynomial and A, B are two matrices in the same orbit [1], then $p(A)$ and $p(B)$ are also in the same orbit. (Therefore, p defines a function on the orbit space.)

(2) Is the converse of the previous problem true? If yes, give a proof. If no, find a counterexample.

(3) **(a)** Let $\mathbb{C}_n[x]$ denote the set of all polynomials with complex coefficients with degree at most n. Find the minimal polynomial of the differentiation operator.

(b) Let $\mathbb{C}[x]$ denote the set of all polynomials with complex coefficients. Does the differentiation operator have a minimal polynomial?

(4) Let V be a finite dimensional vectorspace and T be a linear operator on V. Show that if W is a T-invariant subspace, then the minimal polynomial of T_W divides the minimal polynomial of T.

(5) Let V be a vectorspace and W be a subspace.

(a) Define the quotient vectorspace V/W.

(b) Suppose that V is finite dimensional. What is the dimension of the quotient space?

(c) Give an example of an infinite dimensional vectorspace V and subspaces W_1, W_2 such that V/W_1 is finite dimensional and V/W_2 is infinite dimensional.

(d) Suppose that V' is another vectorspace and $T : V \to V'$ is a linear function. Show that if $W \subseteq \ker(T)$, then T defines a function $V/W \to V'$ by sending $\overline{x}$ to $T(x)$. Discuss why the assumption $W \subseteq \ker(T)$ is crucial.

(e) Let $T : V \to V'$ be a linear map. Show that the induced function $V/N(T) \to V'$ is one-to-one. Deduce that $V/N(T) \cong R(T)$. [2]

(6) Let V be a finite dimensional vectorspace and T be a linear operator on V. Suppose that W is a T-invariant subspace. Let $x \in V$ such that $x \notin W$.

(a) Show that there exists a unique monic polynomial g of least positive degree such that $g(T)(x) \in W_1$. (Hint: Previous question.)

(b) Show that g divides the minimal polynomial of T.

(7) For $n = 1, 2, 3, 4$, pick your favorite linear operator on an n-dimensional vectorspace and find its minimal polynomial.

[1] This is just a fancy way of saying that A and B are similar.

[2] This is called "The Isomorphism Theorem" and it will appear in all algebra courses you take.