The Minimal Polynomial

(1) Let $F$ be an algebraically closed field and consider the conjugation action of $GL(n, F)$ on $M_n(F)$. Show that if $p$ is a polynomial and $A, B$ are two matrices in the same orbit, then $p(A)$ and $p(B)$ are also in the same orbit. (Therefore, $p$ defines a function on the orbit space.)

(2) Is the converse of the previous problem true? If yes, give a proof. If no, find a counterexample.

(3) (a) Let $\mathbb{C}_n[x]$ denote the set of all polynomials with complex coefficients with degree at most $n$. Find the minimal polynomial of the differentiation operator.

(b) Let $\mathbb{C}[x]$ denote the set of all polynomials with complex coefficients. Does the differentiation operator have a minimal polynomial?

(4) Let $V$ be a finite dimensional vectorspace and $T$ be a linear operator on $V$. Show that if $W$ is a $T$-invariant subspace, then the minimal polynomial of $T_W$ divides the minimal polynomial of $T$.

(5) Let $V$ be a vectorspace and $W$ be a subspace.

(a) Define the quotient vectorspace $V/W$.

(b) Suppose that $V$ is finite dimensional. What is the dimension of the quotient space?

(c) Give an example of an infinite dimensional vectorspace $V$ and subspaces $W_1, W_2$ such that $V/W_1$ is finite dimensional and $V/W_2$ is infinite dimensional.

(d) Suppose that $V'$ is another vectorspace and $T : V \to V'$ is a linear function. Show that if $W \subseteq \ker(T)$, then $T$ defines a function $V/W \to V'$ by sending $\bar{x}$ to $T(x)$. Discuss why the assumption $W \subseteq \ker(T)$ is crucial.

(e) Let $T : V \to V'$ be a linear map. Show that the induced function $V/N(T) \to V'$ is one-to-one. Deduce that $V/N(T) \cong R(T)$.  

(6) Let $V$ be a finite dimensional vectorspace and $T$ be a linear operator on $V$. Suppose that $W$ is a $T$-invariant subspace. Let $x \in V$ such that $x \notin W$.

(a) Show that there exists a unique monic polynomial $g$ of least positive degree such that $g(T)(x) \in W_1$. (Hint: Previous question.)

(b) Show that $g$ divides the minimal polynomial of $T$.

(7) For $n = 1, 2, 3, 4$, pick your favorite linear operator on an $n$-dimensional vectorspace and find its minimal polynomial.

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1 This is just a fancy way of saying that $A$ and $B$ are similar.

2 This is called “The Isomorphism Theorem” and it will appear in all algebra courses you take.