More on Vectorspaces

(1) *The following question was on the exam. I wonder how many people realised that there is an issue with this problem.* Let V be a vectorspace over an arbitrary field. If x, y, z are linearly independent vectors in V, show that $x - y, z - x, z + y$ are also linearly independent.

(2) Let V, W are vectorspaces over a field F. We say that a function $f : V \to W$ is *linear* if $f(x + y) = f(x) + f(y)$ for all $x, y \in V$ and $f(cx) = cf(x)$ for all $c \in F, x \in V$.

(a) If $f : V \to W$ is linear, then $f(0) = 0$. Explain why we don't need to specify which 0 we are talking about. There is no ambiguity.

(b) Show that the kernel of f is a subspace of V where the kernel is defined by

$$\ker f = \{x \in V : f(x) = 0\}$$

.

(c) Show that the image of f is a subspace of W where the image is defined by

$$im\ f = \{y \in W : \text{there is } x \in V \text{ s.t. } f(x) = y\}$$

.

(d) Let $\{v_1, \ldots, v_n\}$ be a basis of V. Show that the image of f is spanned by $\{f(v_1), \ldots, f(v_n)\}$.

(e) Show that f is one-to-one if and only if $\ker f = \{0\}$.

(4) Let V be a vectorspace over a field F. Then, the dual of V is defined to be

$$V^* = \{f : V \to F | f \text{ is linear.}\}$$

Ask questions about V^* and answer them. Here are some:

(a) Find a basis for V^*.

(b) What is $\dim V^*$?

(5) Consider a triangle with vertices v_1, v_2, v_3. Define V_1 to be the three dimensional vectorspace with basis v_1, v_2, v_3; V_2 to be three dimensional vectorspace with basis v_1v_2, v_2v_3, v_3v_1 and V_3 to be the one dimensional vectorspace with basis $v_1v_2v_3$. (This is not multiplication. By v_1v_2, I simply mean the edge connecting v_1 to v_2. Similarly, v_1, v_2, v_3 is the triangle itself with its interior.) Define the following linear maps:

$$\begin{aligned} f : V_3 &\to V_2 \\ v_1v_2v_3 &\mapsto v_1v_2 + v_2v_3 + v_3v_1 \end{aligned}$$

and

$$\begin{aligned} g : V_2 &\to V_1 \\ v_1v_2 &\mapsto v_1 - v_2 \\ v_2v_3 &\mapsto v_2 - v_3 \\ v_3v_1 &\mapsto v_3 - v_1 \end{aligned}$$

(Note that we only define the functions on the basis elements. Then we use linearity to define the function on the entire vectorspace.) Show that $g \circ f = 0$. That is, $\mathrm{im} f \subseteq \ker g$. Is it true that $\mathrm{im} f = \ker g$?