More on Vectorspaces

(1) The following question was on the exam. I wonder how many people realised that there is an issue with this problem. Let $V$ be a vectorspace over an arbitrary field. If $x, y, z$ are linearly independent vectors in $V$, show that $x - y, z - x, z + y$ are also linearly independent.

(2) Let $V, W$ are vectorspaces over a field $F$. We say that a function $f : V \to W$ is linear if $f(x + y) = f(x) + f(y)$ for all $x, y \in V$ and $f(cx) = cf(x)$ for all $c \in F, x \in V$.

(a) If $f : V \to W$ is linear, then $f(0) = 0$. Explain why we don’t need to specify which 0 we are talking about. There is no ambiguity.

(b) Show that the kernel of $f$ is a subspace of $V$ where the kernel is defined by

$$\ker f = \{x \in V : f(x) = 0\}$$

(c) Show that the image of $f$ is a subspace of $W$ where the image is defined by

$$\text{im } f = \{y \in W : \text{there is } x \in V \text{ s.t. } f(x) = y\}$$

(d) Let $\{v_1, \ldots, v_n\}$ be a basis of $V$. Show that the image of $f$ is spanned by $\{f(v_1), \ldots, f(v_n)\}$.

(e) Show that $f$ is one-to-one if and only if $\ker f = \{0\}$.

(4) Let $V$ be a vectorspace over a field $F$. Then, the dual of $V$ is defined to be

$$V^* = \{f : V \to F | f \text{ is linear}\}$$

Ask questions about $V^*$ and answer them. Here are some:

(a) Find a basis for $V^*$.

(b) What is dim $V^*$?

(5) Consider a triangle with vertices $v_1, v_2, v_3$. Define $V_1$ to be the three dimensional vectorspace with basis $v_1, v_2, v_3$; $V_2$ to be three dimensional vectorspace with basis $v_1v_2, v_2v_3, v_3v_1$ and $V_3$ to be the one dimensional vectorspace with basis $v_1v_2v_3$. (This is not multiplication. By $v_1v_2$, I simply mean the edge connecting $v_1$ to $v_2$. Similarly, $v_1, v_2, v_3$ is the triangle itself with its interior.) Define the following linear maps:

$$f : V_3 \to V_2$$

$$v_1v_2v_3 \mapsto v_1v_2 + v_2v_3 + v_3v_1$$

and

$$g : V_2 \to V_1$$

$$v_1v_2 \mapsto v_1 - v_2$$

$$v_2v_3 \mapsto v_2 - v_3$$

$$v_3v_1 \mapsto v_3 - v_1$$

(Note that we only define the functions on the basis elements. Then we use linearity to define the function on the entire vectorspace.) Show that $g \circ f = 0$. That is, $\text{im } f \subseteq \ker g$. Is it true that $\text{im } f = \ker g$?