Vectorspaces

(1) Let Fun(\(\mathbb{R}, \mathbb{R}\)) be the vectorspace of all functions from \(\mathbb{R}\) to \(\mathbb{R}\), with operations defined as follows:

(i) For \(f, g \in \text{Fun}(\mathbb{R}, \mathbb{R})\), \(f + g\) is the function defined by the rule \((f + g)(x) := f(x) + g(x)\).

(ii) For \(f \in \text{Fun}(\mathbb{R}, \mathbb{R})\) and \(c \in \mathbb{R}\), \(cf\) is the function defined by the rule \((cf)(x) := cf(x)\).

Show that continuous functions form a subspace of Fun(\(\mathbb{R}, \mathbb{R}\)). How about differentiable functions? Functions with the property \(f \circ f = f\)?

(2) Let \(P_n(\mathbb{R})\) be the vectorspace of all polynomials in degree less than or equal to \(n\) with coefficients from \(\mathbb{R}\). The operations are defined as follows:

(i) \((a_0 + a_1 x + \ldots + a_n x^n) + (b_0 + b_1 x + \ldots + b_n x^n) := a_0 + b_0 + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n\),

(ii) \(c(a_0 + a_1 x + \ldots + a_n x^n) := ca_0 + ca_1 x + \ldots + ca_n x^n\).

Which of the following subsets of \(P_n(\mathbb{R})\) are subspaces?

(a) The subset consisting of all polynomials of degree \(n\).

(b) The subset consisting of all polynomials of degree \(n\) and the zero polynomial.

(c) The subset consisting of all polynomials with \(a_0 = 0\).

(d) The subset consisting of all polynomials with \(a_0 = a_1 = a_2\).

(e) The subset consisting of all polynomials with even degree.

(3) Let \(V\) be a vectorspace and \(u, v \in V\) and \(c, d\) be scalars.

(a) Convince yourself that \(u + v, cu, dv \in V\).

(b) Now convince yourself that \(cu + dv \in V\).

(c) Now convince yourself that any vector you can obtain starting from \(u\) and \(v\) and using two operations must be of the form \(cu + dv\).

(4) Let \(V\) be a vectorspace and \(v_1, \ldots, v_n \in V\). Show that if \(u \in \text{span}\{v_1, \ldots, v_n\}\), then \(\text{span}\{v_1, \ldots, v_n\} = \text{span}\{v_1, \ldots, v_n, u\}\). Is the converse true?

(5) Identify the following statements as (True or False). If it is true, give a short proof. If it is false, give a counterexample. Let \(V\) be a vectorspace, \(W\) be a subspace of \(V\) and \(u, v \in V\).

(a) If \(u + v \in W\), then \(u, v \in W\).

(b) If \(u, v \in W\), then \(\text{span}\{u, v\} \subseteq W\).

(c) Let \(Q \subseteq V \oplus V\) be a subspace of \(V \oplus V\). If \((u, v) \in Q\), then \((v, u) \in Q\).

(d) \(V\) is a subspace of \(V \oplus V\).

(e) \(V\) can be made into a subspace of \(V \oplus V\) in more than one ways.