Vectorspaces

(1) Let $\mathrm{Fun}(\mathbb{R},\mathbb{R})$ be the vectorspace of all functions from $\mathbb{R}$ to $\mathbb{R}$, with operations defined as follows:

(i) For $f, g \in \mathrm{Fun}(\mathbb{R},\mathbb{R})$, $f+g$ is the function defined by the rule $(f+g)(x) := f(x)+g(x)$.

(ii) For $f \in \mathrm{Fun}(\mathbb{R},\mathbb{R})$ and $c \in \mathbb{R}$, cf is the function defined by the rule $(cf)(x) := cf(x)$.

Show that continuous functions form a subspace of $\mathrm{Fun}(\mathbb{R},\mathbb{R})$. How about differentiable functions? Functions with the property $f \circ f = f$?

(2) Let $P_n(\mathbb{R})$ be the vectorspace of all polynomials in degree less than or equal to n with coefficients from $\mathbb{R}$. The operations are defined as follows:

(i) $(a_0+a_1x+\ldots+a_nx^n)+(b_0+b_1x+\ldots+b_nx^n) := a_0+b_0+(a_1+b_1)x+\ldots+(a_n+b_n)x^n$,

(ii) $c(a_0+a_1x+\ldots+a_nx^n) := ca_0+ca_1x+\ldots+ca_nx^n$.

Which of the following subsets of $P_n(\mathbb{R})$ are subspaces?

(a) The subset consisting of all polynomials of degree n.

(b) The subset consisting of all polynomials of degree n and the zero polynomial.

(c) The subset consisting of all polynomials with $a_0 = 0$.

(d) The subset consisting of all polynomials with $a_0 = a_1 = a_2$.

(e) The subset consisting of all polynomials with even degree.

(3) Let V be a vectorspace and $u, v \in V$ and c, d be scalars.

(a) Convince yourself that $u+v, cu, dv \in V$.

(b) Now convince yourself that $cu+dv \in V$.

(c) Now convince yourself that any vector you can obtain starting from u and v and using two operations must be of the form $cu+dv$.

(4) Let V be a vectorspace and $v_1,\ldots,v_n \in V$. Show that if $u \in \mathrm{span}\{v_1,\ldots,v_n\}$, then $\mathrm{span}\{v_1,\ldots,v_n\} = \mathrm{span}\{v_1,\ldots,v_n,u\}$. Is the converse true?

(5) Identify the following statements as **(True or False)**. If it is true, give a short proof. If it is false, give a counterexample. Let V be a vectorspace, W be a subspace of V and $u, v \in V$.

(a) If $u+v \in W$, then $u, v \in W$.

(b) If $u, v \in W$, then $\mathrm{span}\{u,v\} \subseteq W$.

(c) Let $Q \subseteq V \oplus V$ be a subspace of $V \oplus V$. If $(u,v) \in Q$, then $(v,u) \in Q$.

(d) V is a subspace of $V \oplus V$.

(e) V can be made into a subspace of $V \oplus V$ in more than one ways.