Vectorspaces

(1) Let $V = \{x \in \mathbb{R} : x > 0\}$ be the set of positive real numbers. Show that V is a vectorspace over $\mathbb{R}$ with the following two operations:

(i) For any $x, y \in V$, $x \oplus y = xy$,

(ii) For any $c \in \mathbb{R}$ and $x \in V$, $c \cdot x = x^c$.

(2) Let $V = \mathbb{R}^2$.

(a) Show that V is not a vectorspace over $\mathbb{R}$ with the following operations:

(i) For any $x = (x_1, x_2), y = (y_1, y_2) \in V$, $x \oplus y = (\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2})$ (observe that this is the midpoint of x and y),

(ii) For any $c \in \mathbb{R}$ and $x = (x_1, x_2) \in V$, $c \cdot x = (cx_1, cx_2)$

(b) Consider the following two statements:

(i) For all $x \in V$, there exists an $e \in V$ such that $e + x = x$.

(ii) There exists an $e \in V$ such that for all $x \in V, e + x = x$.

Show that the first statement is true while the second one is false. Discuss the existence of a zero vector.

(3) Let V be a vectorspace over a field F.

(a) Show that the zero vector is unique.

(b) Show that additive inverses are unique.

(c) Show that for any $x \in V$, $0 \cdot x = 0$. (Note that the first 0 is the zero element of the field and the second one is the zero vector of the vectorspace. But we don't need to specify this. Why is it clear from the context?)

(d) Show that for all $x \in V$, $(-1)x = -x$ where $-x$ is the additive inverse of x.

(e) Show that for any $c \in F$, we have $c \cdot 0 = 0$.

(f) Show that for all $x \neq 0 \in V$, if $c \neq 0$, then $c \cdot x \neq 0$.

(g) Show that for all $x \neq 0$, if $a \neq b$, then $a \cdot x \neq b \cdot x$.