Determinants - And a review of Dual Spaces

(1) There was a request for a review of dual spaces in the Wednesday tutorial. Solve the following problems. Most answers are one or two lines. F will always denote a field and all vectorspaces are over F.

(a) Show that F is a 1-dimensional vectorspace by addition and scalar multiplication defined by the most obvious way.

(b) Show that if V and W are vectorspaces, then $L(V,W) = \{f : V \to W : f \text{ is linear}\}$ is a vectorspace with the following operations: $(f+g)(v) = f(v)+g(v)$ and $(cf)(v) = cf(v)$. Make sure you understand how this definition works.

(c) Conclude that $L(V,F)$ is a vectorspace. We call this vectorspace the dual space. V^*.

(d) Recall that if V is finite dimensional with a basis $\{v_1, \ldots, v_n\}$, then any element $v \in V$ can be written as $v = a_1v_1 + \ldots + a_nv_n$ for some $a_1, \ldots, a_n \in F$. If $f \in L(V,W)$, then show that $f(v) = a_1f(v_1) + \ldots a_nf(v_n)$ and conclude that to define a linear function, it is enough to define its values at a basis.

(e) Define $v_i^* : V \to F$ to be the linear function defined on the basis $\{v_1 \ldots, v_n\}$ as follows:

$$v_i^*(v_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Show that for $v = a_1v_1 + \ldots + a_n$, we have $v_i^*(v) = a_i$.

(f) Use parts (d) and (e) to conclude that if $f \in V^*$, then for any $v \in V$, we have

$$f(v) = v_1^*(v)f(v_1) + \ldots + v_n^*(v)f(v_n)$$

(g) Explain why this gives us $f(v) = f(v_1)v_1^*(v) + \ldots + f(v_n)v_n^*(v)$.

(h) Letting $f(v_i) = c_i$, write $f(v) = c_1v_1^*(v) + \ldots + c_nv_n^*(v)$. Using the definition in part (b), show that this means $f(v) = [c_1v_1^* + \ldots + c_nv_n^*](v)$.

(i) Recall that two functions $f_1, f_2 : X \to Y$ are equal if $f_1(x) = f_2(x)$ for all $x \in X$. Conclude from part (h) that since v was arbitrary, we have

$$f = c_1v_1^* + \ldots + c_nv_n^* = f(v_1)v_1^*(v) + \ldots + f(v_n)v_n^*(v)$$

(j) Conclude that $f \in span\{v_1^*, \ldots, v_n^*\}$ and that since f was arbitrary $span\{v_1^*, \ldots, v_n^*\} = V^*$.

(k) Recall from parts (g) and (h) that $f(v) = f(v_1)v_1^*(v) + \ldots + f(v_n)v_n^*(v)$. Suppose that $c_1v_1^* + \ldots + c_nv_n^* = 0 \in V^*$ and so that for all $v \in V$, we have $c_1v_1^*(v) + \ldots + c_nv_n^*(v) = 0$. Since this is true for all v, it must be true for $v_1, \ldots, v_n$. Conclude that $c_1 = \ldots = c_n = 0$. Conclude that $\{v_1^*, \ldots, v_n^*\}$ is a linearly independent set.

(l) Conclude that $\{v_1^*, \ldots, v_n^*\}$ is a basis for V^*.

(2) Let $V = P_2(\mathbb{R})$ and consider the basis $\{1, x, x^2\}$. For convenience, call $q_0 = 1, q_1 = x, q_2 = x^2$.

(a) Define $ev_5 : V \to \mathbb{R}$ by the formula $ev_5(p) = p(5)$. (ev stands for "evaluation.") Show that $ev_5(x^2+3) = 28$, $ev_5(16) = 16$, $ev_5(9+x) = 14$. Make sure you understand what this function ev_5 does.

(b) Show that $ev_5 \in V^*$ (you need to check that it is linear).

(c) Using question 1(i) write ev_5 as a linear combination of q_0^*, q_1^* and q_2^*.

(3) Let V be a finite dimensional vectorspace with basis $\{v_1, \ldots, v_n\}$. Define $T : V \to V^{**}$ to be $T(v) = ev_v$ where $ev_v : V^* \to F$ is the function $ev_v(f) = f(v)$.

(a) Let $v = a_1v_1 + \ldots + a_nv_n$. Show that $T(v) = 0$ if and only if $f(v) = 0$ for all $f \in V^*$.

(b) Show that this implies $v = 0$. (Hint: Question 1(e)).

(c) Conclude that T is one-to-one.

(d) Use question 1 to show that V^* is n-dimensional. Use question 1 again to show that V^{**} is also n-dimensional.

(e) Conclude that T has to be an isomorphism.

(4) We say that a square matrix A is *nilpotent* if there is some $n \geq 0$ such that $A^n = 0$.

(a) Give a non-zero example of a nilpotent matrix.

(b) Give an example of a nilpotent matrix A such that $A \neq 0$ but $A^2 = 0$.

(c) Give an example of a nilpotent matrix A such that $A^2 \neq 0$ but $A^3 = 0$.

(d) Give an example of a nilpotent matrix A such that $A^3 \neq 0$ but $A^4 = 0$.

(e) Let $p_n = x^n$. Show that the definition of a nilpotent matrix says that A is a nilpotent matrix if and only if $p_n(A) = 0$ for some $n \geq 0$.

(f) Show that if A is nilpotent, then $\det(A) = 0$.

(g) **True or False.** If A, B are nilpotent matrices of the same size, then $A + B$ is also nilpotent.

(h) **True or False.** If A, B are nilpotent matrices of the same size, then AB is also nilpotent.

(5) We say that a square matrix is idempotent if $A^2 = A$.

(a) Show that there is only one invertible idempotent $n \times n$ matrix.

(b) Give an example of a singular (non-invertible) idempotent matrix. (Think of projections.)

(6) **True or False.** Suppose that A is a square matrix and $p(A) = 0$ for some polynomial p. Let $c = \det(A) \in F$. Then, $p(c) = 0$ as well.