Determinants - And a review of Dual Spaces

(1) There was a request for a review of dual spaces in the Wednesday tutorial. Solve the following problems. Most answers are one or two lines. $F$ will always denote a field and all vectorspaces are over $F$.

(a) Show that $F$ is a 1-dimensional vectorspace by addition and scalar multiplication defined by the most obvious way.

(b) Show that if $V$ and $W$ are vectorspaces, then $L(V, W) = \{ f : V \to W : f \text{ is linear} \}$ is a vectorspace with the following operations: $(f + g)(v) = f(v) + g(v)$ and $(cf)(v) = cf(v)$. Make sure you understand how this definition works.

(c) Conclude that $L(V, F)$ is a vectorspace. We call this vectorspace the dual space. $V^*$.

(d) Recall the definition in part (b), show that this means $L(V, F)$ is finite dimensional with a basis $\{ v_1, \ldots, v_n \}$, then any element $v \in V$ can be written as $v = a_1 v_1 + \ldots + a_n v_n$ for some $a_1, \ldots, a_n \in F$. If $f \in L(V, W)$, then show that $f(v) = a_1 f(v_1) + \ldots + a_n f(v_n)$ and conclude that $f$ is a linear function, it is enough to define its values at a basis.

(e) Define $v^*_i : V \to F$ to be the linear function defined on the basis $\{ v_1, \ldots, v_n \}$ as follows:

$$v^*_i(v_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Show that for $v = a_1 v_1 + \ldots + a_n$, we have $v^*_i(v) = a_i$.

(f) Use parts (d) and (e) to conclude that if $f \in V^*$, then for any $v \in V$, we have

$$f(v) = v^*_1(v) f(v_1) + \ldots + v^*_n(v) f(v_n)$$

(g) Explain why this gives us $f(v) = f(v_1) v^*_1(v) + \ldots + f(v_n) v^*_n(v)$.

(h) Letting $f(v_i) = c_i$, write $f(v) = c_1 v^*_1(v) + \ldots + c_n v^*_n(v)$. Using the definition in part (b), show that this means $f(v) = [c_1 v^*_1 + \ldots + c_n v^*_n](v)$.

(i) Recall that two functions $f_1, f_2 : X \to Y$ are equal if $f_1(x) = f_2(x)$ for all $x \in X$. Conclude from part (h) that since $v$ was arbitrary, we have

$$f = c_1 v^*_1 + \ldots + c_n v^*_n = f(v_1) v^*_1(v) + \ldots + f(v_n) v^*_n(v)$$

(j) Conclude that $f \in \text{span}\{v^*_1, \ldots, v^*_n\}$ and that since $f$ was arbitrary $\text{span}\{v^*_1, \ldots, v^*_n\} = V^*$.

(k) Recall from parts (g) and (h) that $f(v) = f(v_1) v^*_1(v) + \ldots + f(v_n) v^*_n(v)$. Suppose that $c_1 v^*_1 + \ldots + c_n v^*_n = 0 \in V^*$ and so that for all $v \in V$, we have $c_1 v^*_1(v) + \ldots + c_n v^*_n(v) = 0$. Since this is true for all $v$, it must be true for $v_1, \ldots, v_n$. Conclude that $c_1 = \ldots = c_n = 0$. Conclude that $\{v^*_1, \ldots, v^*_n\}$ is a linearly independent set.

(l) Conclude that $\{v^*_1, \ldots, v^*_n\}$ is a basis for $V^*$.

(2) Let $V = P_2(\mathbb{R})$ and consider the basis $\{1, x, x^2\}$. For convenience, call $q_0 = 1, q_1 = x, q_2 = x^2$.

(a) Define $ev_5 : V \to \mathbb{R}$ by the formula $ev_5(p) = p(5)$. ($ev$ stands for “evaluation.”) Show that $ev_5(x^2 + 3) = 28$, $ev_5(16) = 16$, $ev_5(9 + x) = 14$. Make sure you understand what this function $ev_5$ does.
(b) Show that \( ev_5 \in V^* \) (you need to check that it is linear).
(c) Using question 1(i) write \( ev_5 \) as a linear combination of \( q_0^*, q_1^* \) and \( q_2^* \).

(3) Let \( V \) be a finite dimensional vectorspace with basis \( \{v_1, \ldots, v_n\} \). Define \( T : V \to V^{**} \) to be \( T(v) = ev_v \) where \( ev_v : V^* \to F \) is the function \( ev_v(f) = f(v) \).

(a) Let \( v = a_1 v_1 + \ldots + a_n v_n \). Show that \( T(v) = 0 \) if and only if \( f(v) = 0 \) for all \( f \in V^* \).
(b) Show that this implies \( v = 0 \). (Hint: Question 1(e)).
(c) Conclude that \( T \) is one-to-one.
(d) Use question 1 to show that \( V^* \) is \( n \)-dimensional. Use question 1 again to show that \( V^{**} \) is also \( n \)-dimensional.
(e) Conclude that \( T \) has to be an isomorphism.

(4) We say that a square matrix \( A \) is **nilpotent** if there is some \( n \geq 0 \) such that \( A^n = 0 \).

(a) Give a non-zero example of a nilpotent matrix.
(b) Give an example of a nilpotent matrix \( A \) such that \( A \neq 0 \) but \( A^2 = 0 \).
(c) Give an example of a nilpotent matrix \( A \) such that \( A^2 \neq 0 \) but \( A^3 = 0 \).
(d) Give an example of a nilpotent matrix \( A \) such that \( A^3 \neq 0 \) but \( A^4 = 0 \).
(e) Let \( p_n = x^n \). Show that the definition of a nilpotent matrix says that \( A \) is a nilpotent matrix if and only if \( p_n(A) = 0 \) for some \( n \geq 0 \).
(f) Show that if \( A \) is nilpotent, then \( \det(A) = 0 \).
(g) **True or False.** If \( A, B \) are nilpotent matrices of the same size, then \( A + B \) is also nilpotent.
(h) **True or False.** If \( A, B \) are nilpotent matrices of the same size, then \( AB \) is also nilpotent.

(5) We say that a square matrix is **idempotent** if \( A^2 = A \).

(a) Show that there is only one invertible idempotent \( n \times n \) matrix.
(b) Give an example of a singular (non-invertible) idempotent matrix. (Think of projections.)

(6) **True or False.** Suppose that \( A \) is a square matrix and \( p(A) = 0 \) for some polynomial \( p \). Let \( c = \det(A) \in F \). Then, \( p(c) = 0 \) as well.