Normal and Self-Adjoint Operators

(1) The following question is from your assignment: Assume that $T$ is a linear operator on a finite dimensional complex inner product space $V$ with an adjoint $T$. Show that if $T$ satisfies $\langle T(x), x \rangle = 0$ for all $x \in V$, then $T$ is the zero transformation.

*Remark.* Before the midterm, we had seen that this is not the case in a real inner product space.

I will present a wrong proof. There is a mistake in the proof. Identify the mistake (there is only one mistake).

- Let $p_T$ be the characteristic polynomial of $T$. Since we are in a complex vector space, $p_T$ splits. Therefore, $p_T$ has $n$ eigenvalues (counting with multiplicities) where $n = \dim V$.
- Let $\lambda$ be an eigenvalue and $v$ be a corresponding eigenvector. Then, we have
  $$0 = \langle T(v), v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle$$
  and thus $\lambda \langle v, v \rangle = 0$.
- We know that $v$ is nonzero. Hence, $\langle v, v \rangle \neq 0$.
- Thus, $\lambda = 0$. $\lambda$ was an arbitrary eigenvalue, hence all eigenvalues are zeroes.
- Since all eigenvalues of $T$ are zeros, $T$ is the zero vector.

(2) Label the following statements as true or false. Justify your answer.

(a) Let $T$ be a linear operator on a finite dimensional complex inner product space $V$. Then, there is an orthonormal basis $\beta$ of $V$ such that $[T]_\beta$ is diagonal.

(b) Let $T$ be a linear operator on a finite dimensional complex inner product space $V$. Then, there is an orthonormal basis $\beta$ of $V$ such that $[T]_\beta$ is upper triangular.

(c) Let $T$ be a linear operator on a finite dimensional inner product space $V$. If $p_T$ splits, there is an orthonormal basis $\beta$ of $V$ such that $[T]_\beta$ is upper triangular.

(d) Let $T$ be a linear operator on a finite dimensional inner product space $V$. If $T$ is diagonalizable and $v, w$ are linearly independent eigenvectors, then $v, w$ are orthogonal.

(e) Let $T$ be a linear operator on a finite dimensional complex inner product space $V$ and suppose that there is an orthonormal basis $\beta$ of $V$ such that $[T]_\beta$ is diagonal. Then, $[T]_\beta^*$ is also diagonal.

(f) Let $T$ be a diagonalizable linear operator on a finite dimensional complex inner product space $V$. Then $TT^* = T^* T$.

(g) Let $T$ be a linear operator on a finite dimensional complex inner product space $V$ and $T$ has $n$ orthogonal eigenvectors. Then $TT^* = T^* T$.

(h) If $A$ is a normal matrix with complex entries, then $A$ is self-adjoint.

(i) If $A$ is a normal matrix with complex entries, then $A$ is symmetric.

(j) If $A$ is a normal matrix with complex entries, then $A$ is either symmetric or skew-symmetric.