A couple of important things:

- Be careful with quantifiers. Most of you forgot them in Assignment 3. I did not cut marks this time, but this is a very VERY important issue.
- Words of encouragement: Check this out before the midterm http://www.math.vanderbilt.edu/~msapir/tt.html. I suggest you listen to this in a quiet place, preferably in the dark, at least three times a day.

(1) Let $k = \mathbb{C}$ or $\mathbb{R}$. Let $V = M_2(k)$ be the space of $2 \times 2$ matrices with entries from $k$. $V$ has a natural inner product space structure given by $\langle A, B \rangle = \text{tr}(B^*A)$ for $A, B \in V$.

(a) (Unimportant question.) Why is this inner product "natural"? What is the reason behind defining it like this?

(b) We say that a matrix $A$ is self-adjoint if $A^* = A$. Show that the set $S$ of all self-adjoint $2 \times 2$ matrices form a subspace of $V$. Find the dimension of $S$. (You already did this exercise several times in the first half of this course, when $k = \mathbb{R}$.) Important Note: In tutorial, we quickly realized that if the underlying field is $\mathbb{C}$, $S$ is not a subspace. Do this exercise over $\mathbb{R}$.

(c) Find an orthonormal basis of $S$.

(d) Let $Q$ be the set of all matrices in $V$ with the following property:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} b & d \\ a & c \end{bmatrix}$$

Show that $Q$ is a subspace and find a basis for $Q^\perp$.

(2) Label the following statements as true or false and justify your answer.

(a) Let $V$ be a finite dimensional inner product space and $W$ be a subspace of $V$. If $V = W \oplus U$ for a subspace $U$ of $W$, then $U$ is the orthogonal complement of $W$.

(b) Let $V$ be a vector space and $W_1, W_2, W_3$ be subspaces of $V$. If $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_3$, then we have $W_2 = W_3$.

(c) Let $V$ be a finite dimensional vector space and $U, W$ be subspaces of $V$. Let $\alpha, \beta$ be bases for $U$ and $W$, respectively. If $|\alpha \cup \beta| = \dim V$, then $V = U \oplus W$.

(d) Let $V$ be a finite dimensional vector space and $U, W$ be subspaces of $V$. Let $\alpha, \beta$ be bases for $U$ and $W$, respectively. If $|\alpha \cup \beta| = \dim V$ and $\alpha \cup \beta$ is linearly independent, then $V = U \oplus W$.

(e) Let $V$ be a finite dimensional inner product space and $T : V \to V$ be a linear operator. If $\langle T(x), x \rangle = 0$ for all $x \in V$, then $T$ is the zero transformation.

(f) Let $V$ be a finite dimensional inner product space and $T : V \to V$ be a linear operator. If $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$, then $T$ is invertible.

(g) Let $V$ be a finite dimensional vector space and $T : V \to V$ be a linear operator. Then, $V = \ker T \oplus \text{im} T$.

(h) Let $V$ be a finite dimensional inner product space and $T : V \to V$ be a linear operator. Let $\beta$ be a basis for $V$. If $[T]_\beta$ is a self-adjoint matrix, then $T$ is a self-adjoint transformation.