Comments

These weekly questions are designed for you to review the material we cover in the class, or to catch up if you miss a lecture. At the end of each of each weekly questions, you will see questions assigned from the textbook. You are strongly advised to attempt all of these questions. If English isn't your first language, you are encouraged to solve the questions in the language you are most comfortable with. You may then translate your answer to English. If there is anything that you don't understand, you can always ask to

- your instructor in office hours.
- your TA in tutorials.
- general math TAs in Math Aid Centre.

It is perfectly fine to be confused. These resources are there to help you. This is 8 hours of help per week. For more information on these resources, see the syllabus. In addition to these, you are very strongly encouraged to meet at least a couple of fellow students in your class. Even better, create your own study group. Mathematics is usually easier and definitely more fun when shared with others. You will not believe how much one can learn from different people's perspectives. Sometimes, one small comment from a friend can make an entire confusing section crystal clear. Sometimes, you can realize that the section you thought you understand is not actually well understood when you see you can't explain the first definition to your friends. This is a small class and you can always find a coffee shop or an empty classroom to fit all of your study group.

A Review of Functions

(1) When we have a function $f : X \to Y$, we say that X is the **domain** of the function. In this course, we are dealing with functions whose domains are subsets of real numbers. For this reason, when we say "find the domain of a function", it will usually mean "find the largest possible domain for which this function can be defined". For instance, we say that the domain of the function $\frac{1}{x}$ is $\mathbb{R} \setminus \{0\}$. This means that this function is defined for all real values of x except zero. Find the domain of the following functions:

(a) $\sqrt{x}$

(b) $\sqrt{x-1}$

(c) $\sqrt{x^2-1}$

(d) $\sqrt{x^2+1}$

(e) $\dfrac{1}{x}$

(f) $\dfrac{1}{x-3}$

(g) $\dfrac{1}{x^2-4}$

(h) $\dfrac{1}{x^2-3}$

(g) $\dfrac{1}{x^2+3}$

(h) $\dfrac{2}{\sin(x)}$

(i) $\dfrac{4}{\sin(x)-1}$

(j) $\dfrac{4}{\sin(x)-2}$

(2) When we define a function $f : X \to Y$, we say that Y is the **codomain** of the function. This gives us an information about where the output of the function lives. However, sometimes the codomain may be unnecessarily big. This is similar to answering "I am from the solar system" to the question "Where are you?". Instead of the information about "where the output lives", we therefore ask "which real numbers can be outputs". This tells us the exact place where the output lives, not more than that. We usually say that this is the **range** of the function. More precisely, the range of a function $f : X \to Y$ is the subset of Y defined as $\{y \in Y : f(a) = y \text{ for some } x\}$. That is, the range of a function is the collection of elements in the codomain which are targeted by the function. For instance, if we have a constant function defined by $f(x) = 3$, the range of f is $\{3\}$. Because this function only hits this point. Similarly, the function $\sin(x)$ has range $[-1, 1]$ as we discussed last week. Find the range of the following functions (assuming that they are defined on the largest domain possible):

(a) $\cos(x)$

(b) $\sin(x) - 21$

(c) $\sin^2(x)$

(d) $2x + 4$

(e) $x^2 + 1$

(f) x^3

(g) $\sqrt{x}$

(h) 4^x

(i) $|x|$

(j) $|1 - x^2|$

(j) $|x^2 - 2|$

(3) You are not required in this course to answer the previous questions in mathematical notation. But you should definitely learn how to do so. Otherwise, you will have troubles in higher level math classes. For instance, in this course, you can say "the set of all positive real numbers except two". But you should learn to write this set as $\{x \in \mathbb{R} \text{ such that } x > 0\} \setminus \{2\}$ or $(0, 2) \cup (2, \infty)$. Both during the lectures and during the tutorials we will use notations like this. We will also use these notations in the tests. And you are not required to answer in these notations, but you are required to understand them. So, start practicing with the previous two questions.

(4) When a function $f : X \to Y$ hits every point in the codomain, that is when the range is equal to the codomain, we say that this function is onto (or some people say *surjective*). For instance any linear function $ax + b$ with $a \neq 0$ is onto. Choose your favorite function, and decide if they are onto. Notice that there is a subtle issue here: If we define the cosine function as $\cos : \mathbb{R} \to \mathbb{R}$, then this is not an onto function. If we define it as $\cos : \mathbb{R} \to [-1, 1]$, then it is onto. Therefore, we should be careful. We should ask "What is the domain and codomain of this function!". If the domain is not given, we will assume it is the largest possible domain. And if the codomain is not given, we will assume it is all of $\mathbb{R}$. So, when we say $2x + 1$ is an onto function, that assume that the domain is $\mathbb{R}$ and the codomain is also $\mathbb{R}$. Make sure that you understand this subtlety.

(5) Recall that when we define a function, we have to make sure that for each possible input, there is one and only one output. That is, an element of the domain can not be mapped to more than one elements of the codomain. However, we may have situations where more than one input give the same output. For instance, if you take a constant function, then every input gives the same output. Another example is the absolute value function. We can see that the distance between -1 and 0 is one unit which is equal to the distance between 0 and 1. Therefore, $|-1| = |1|$. However, if we consider the function $f(x) = 3x + 5$, we see that whenever we give two different inputs, we get different outputs. Indeed, if for two inputs a, b in the domain we get the same output (that is, if $3a + 5 = 3b + 5$), then we have $3a = 3b$ and therefore $a = b$. That is, $f(a) = f(b)$ implies $a = b$ in this case. When this happens, we say that the function is one-to-one. More precisely, we say that a function $f : X \to Y$ is one-to-one if $f(a) = f(b)$ implies $a = b$ for any $a, b \in X$. This may seem complicated at the beginning but it really only tells you that two different elements can not be sent to the same value. Now, choose your favourite functions and decide if they are one-to-one.

(6) It is always good practice to create your own examples. Without looking at any other sources, give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is

(a) Onto but not one-to-one

(b) Neither onto nor one-to-one

(c) Onto and one-to-one

(d) Not onto but one-to-one

Try to do this example in two ways: (a) Drawing a picture (that is, a graph of such function) and (b) Giving the rule of a function.

(7) Is $x + |x|$ an injective function? What are the values of this function for $x = 0$ and $x = -1$? Draw the graph of this function. Now, go to wolframalpha.com and ask "is $x+|x|$ injective?" What does it say? Is it the same as your answer? If not, do you believe wolframalpha or to your own knowledge? In the future, when you need calculus, you will mostly use computer programs designed for your purposes. You will use these programs, so you may ask yourself "Why do I need to learn this? I will just use a computer." Well, sometimes computers are wrong or sometimes when we type in our input, we make a typo. You should at least be able to tell when something is obviously wrong so that you can either correct your typo or change your computer program.

(8) Suppose we have a function $f : X \to Y$.

- If this function is not onto, this means that there is an element $u \in Y$ for which is not hit by any element of X. So, there is no way of sending it back to wherever it came from.
- If this function is not one-to-one, there is an element $u \in Y$ and at least two different elements $x_1, x_2 \in X$ such that $f(x_1) = f(x_2) = u$. So, again we can not send u back to wherever it came from because there are more than one places.
- However, if f is both one-to-one and onto, we can send any $u \in Y$ back to the only place where it came from. That is, there is one and only one element $e \in X$ such that $f(e) = u$. So, I can send u to e. And this defines a function!

Whenever, we have a function which is one-to-one and onto, we can create another function which undoes what the original function does. This new function is called "the inverse" of f, we denote the inverse of f by f^{-1}. And if f has an inverse (if it is one-to-one and onto), we say that f is invertible. For instance, the function $g(x) = \dfrac{x}{2}$ is invertible. And the inverse function is $g^{-1} = 2x$. Find the inverses of the following functions. Be careful with the domains and codomains:

(a) $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = 2x + 3$

(b) $g : [0, \infty) \to [0, \infty), \quad g(x) = x^2$

When we don't give the domain and the codomain and ask you to find the inverse, we mean that the domain is the biggest possible domain and the codomain is the range. (Does this make sense? If not, read again, and think about it.) Find the inverse of the following functions:

(c) $h(x) = \dfrac{4x - 1}{2x + 3}$

(d) $f(x) = \sqrt{1 - x^2}$ with $0 \leq x \leq 1$.

(9) Explain why the function $P : \mathbb{R} \to (0, \infty]$ defined by $P(x) = 10^x$ has an inverse. We give a special name to the inverse of this function. Do you remember what it is?

(10) Explain why the function $\sin : [0, 2\pi) \to [-1, 1]$ does not have an inverse. But $\sin : [-\pi/2, \pi/2] \to [-1, 1]$ has an inverse. We give a special name to it. What is it? How about $\cos : [0, \pi] \to [-1, 1]$ and $\tan : (\dfrac{-\pi}{2}, \dfrac{\pi}{2}) \to \mathbb{R}$?

(11) **More ways to construct new functions from old.** Let A be a subset of real numbers and $f, g : A \to \mathbb{R}$ be two real-valued functions defined on this subset. We can define a new function ()which we call the sum of two functions and denote by $f + g$) with the following formula: Recall that to describe a function, you need to specify its value at every value in the domain. So, we define the value $(f + g)(x)$ as $f(x) + g(x)$.

(a) Let $f(x) = x^2$ and $g(x) = \log_3(x)$. Then, $f + g$ is the function defined by the formula $(f + g)(x) = f(x) + g(x) = x^2 + \log_3(x)$. Calculate $(f + g)(9)$.

We can also define the product function $(fg)(x)$ by the formula $(fg)(x) = f(x)g(x)$. So this new function, first calculates $f(x)$ and $g(x)$ separately and then multiplies their values.

(b) Let $f(x) = 2^x$ and $g(x) = \frac{x^2+1}{x^3+3}$. Calculate $(fg)(3)$.

Finally, a great way to create new functions from old is to consider composition of two functions. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions. Then, the composition $f \circ g$ of f and g is the function "which does g first and then applies f to the output of g". We actually saw some examples of it earlier.

(c) Let $f(x) = x^2$ and $g(x) = x - 2$. Then, $(f \circ g)(x) = f(g(x)) = (x - 2)^2$. Calculate $f \circ g(2)$.