Comments

Congratulations. With today's lecture we are ending the first part of the course. Next week, we will start derivatives.

Intermediate Value Theorem - Final words on limits

(1) Recall from Tuesday's lecture: Suppose that f is continuous on the closed interval $[a, b]$ and let K be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there is a number c in (a, b) such that $f(c) = K$.
A very nice consequence of this is: Suppose that you have a continuous function f on an interval $[a, b]$. Suppose that $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$. Then, there is a number c between a and b such that $f(c) = 0$.

- **(a)** Draw some graphs of continuous functions on the interval $[0, 1]$ such that the value at 0 is negative and the value at 1 is positive. Try and see that it is impossible to draw a graph with above conditions which doesn't cross the horizontal axis!
- **(b)** Let's see this on an example. [Before doing this example, watch this video: https://youtu.be/5Px6fajpSio by the YouTuber singingbanana.] Let's assume that Earth is a sphere. We say that two points on Earth are antipodal if the line segment which connects them passes through the center of the sphere. These are "opposite points" on Earth. Assuming that temperature varies continuously, we can show that at any given time there are two antipodal points on Earth which have the exact same temperature. In fact, we can show that we can find such points on the Equator! Here is how:
  1. Identify the Equator with a circle. In fact, you can think it as the unit circle in the plane.
  2. Consider the function U defined on the interval $[0, 2\pi]$ which takes a number a to the point $(\cos(a), \sin(a))$ on the unit circle. (Recall this is how we defined cos and sin!).
  3. Observe that if two points on the unit circle are antipodal, then their angles differ by π radians. Indeed, by definition of antipodal points, we have a straight angle. This means that, for any $x \in [0, \pi]$, the couple $U(x)$ and $U(x + \pi)$ is an antipodal couple.
  4. Recall that $\sin(0) = \sin(2\pi)$ and $\cos(0) = \cos(2\pi)$. Thus, $U(0) = U(2\pi)$.
  5. Now consider the function T which takes a point on the unit circle to the temperature at the corresponding point on the Equator.
  6. If we first apply U and then T, this means that we are taking a number in $[0, 2\pi]$ to the corresponding real number (temperature). So, we have a function $T \circ U$ from $[0, 2\pi]$ to real numbers. And it is continuous. Observe that $T \circ U(0) = T \circ U(2\pi)$. Indeed, we are talking about the temperature at the same point!
  7. We would like to show that there is a couple of antipodal points which have the same temperature. We know that antipodal couples are of the form $U(x)$ and $U(x + \pi)$. So, our purpose is to show that there is a point $x \in [0, \pi]$ such that $T \circ U(x) = T \circ U(x + \pi)$. Or written in other words, we want to show that there is a point $x \in [0, \pi]$ such that

$$T \circ U(x) - T \circ U(x + \pi) = 0$$

8. So, we consider a new function H defined by the rule

$$H(x) = T \circ U(x) - T \circ U(x + \pi)$$

on $[0, \pi]$. We said that $T \circ U$ is continuous already. We also know that the linear function $g(x) = x + \pi$ is continuous. Therefore, the function which takes x to $T \circ U(x + \pi)$ is also continuous! Because it is a composition of continuous functions (we first apply the linear function, then apply $T \circ U$).

9. Then, we can use the fact that sum of two continuous functions is continuous to conclude that H is a continuous function. Let's rewrite our purpose: Our purpose is to show that there is a number $c \in [0, \pi]$ such that $H(c) = 0$.

10. We compute the following two values:

$$\begin{aligned} H(0) &= T \circ U(0) - T \circ U(\pi) \\ H(\pi) &= T \circ U(\pi) - T \circ U(2\pi) \end{aligned}$$

BUT!! Look at the last sentence of number 6! We have $T \circ U(0) = T \circ U(2\pi)$, so we actually have

$$\begin{aligned} H(0) &= T \circ U(0) - T \circ U(\pi) \\ H(\pi) &= T \circ U(\pi) - T \circ U(2\pi) = T \circ U(\pi) - T \circ U(0) = -H(0) \end{aligned}$$

This tells us that $H(\pi) = -H(0)$. This means
- EITHER $H(0)$ and $H(\pi)$ have different signs,
- OR $H(0) = H(\pi) = 0$

In the first case, Intermediate Value Theorem tells us that there is c in $[0, \pi]$ such that $H(c) = 0$. In the second case, we are already done! Going back, we see that this is exactly what we wanted. So, we showed that there is a pair of antipodal points in the Equator which have the same temperature.

(2) Calculate

$$\lim_{x \to \infty} \sqrt{9x^2 + x} - 3x$$

(3) Calculate

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 3} - \sqrt{3}}{x}$$

(4) Calculate

$$\lim_{x \to \infty} \frac{\sqrt{x^2 + 3} - \sqrt{3}}{x}$$