Comments

These weekly questions are designed for you to review the material we cover in the class, or to catch up if you miss a lecture. At the end of each of each weekly questions, you will see questions assigned from the textbook. You are strongly advised to attempt all of these questions. If English isn't your first language, you are encouraged to solve the questions in the language you are most comfortable with. You may then translate your answer to English. If there is anything that you don't understand, you can always ask to

- your instructor in office hours.
- your TA in tutorials.
- general math TAs in Math Aid Centre.

It is perfectly fine to be confused. These resources are there to help you. This is 12 hours of help per week. For more information on these resources, see the syllabus. In addition to these, you are very strongly encouraged to meet at least a couple of fellow students in your class. Even better, create your own study group. Mathematics is usually easier and definitely more fun when shared with others. You will not believe how much one can learn from different people's perspectives. Sometimes, one small comment from a friend can make an entire confusing section crystal clear. Sometimes, you can realize that the section you thought you understand is not actually well understood when you see you can't explain the first definition to your friends. This is a small class and you can always find a coffee shop or an empty classroom to fit all of your study group.

Continuity

(1) We say that a function f is continuous at a point a if we have

$$\lim_{x\to a} f(x) = f(a)$$

Notice that here, we require three things:

- The function is defined at a.
- The limit $\lim_{x\to a} f(x)$ exists.
- The limit is equal to the value $f(a)$.

In order to be continuous at a point a, the function has to satisfy all of these three properties.

(a) Consider the function F given by the rule

$$F(t) = \frac{t^2-1}{t-1}$$

We know that

$$\lim_{t\to 1} \frac{t^2-1}{t-1} = \lim_{t\to 1} \frac{(t-1)(t+1)}{t-1} = \lim_{t\to 1} t+1 = 2$$

So, the limit exists. However, this function is not continuous at 1 because it is not defined there.

(b) Consider the function G defined by the rule

$$G(x) = \begin{cases} x+1, & \text{if } x < 2, \\ 5, & \text{if } x = 2, \\ x+1, & \text{if } x > 2 \end{cases}$$

This function is defined at 2, moreover, the limit exists at 2:

$$\lim_{x\to 2^-} G(x) = \lim_{x\to 2^+} G(x) = 3 = \lim_{x\to 2}$$

However, we have that the limit is not equal to the value at 2:

$$\lim_{x\to 2} G(x) = 3 \quad \text{and} \quad G(2) = 5$$

So, the function is not continuous at 2.

(c) Consider the function H defined by the rule

$$H(x) = \begin{cases} x+1, & \text{if } x < 2, \\ 5, & \text{if } x = 2, \\ x+3, & \text{if } x > 2 \end{cases}$$

Here, we have

$$\lim_{x\to 2^-} H(x) = 3, \quad \lim_{x\to 2^+} H(x) = 5$$

So, the limit does not exist. Therefore, this function is not continuous at 2.

We will say that f is a continuous function if f is continuous everywhere. We will say that f is continuous at an interval $[a, b]$ if it is continuous at all points in the interval.

(2) Again, we have some basic facts which we can use to say more general things:

(A) The identity function $f(x) = x$ is continuous. This is easy to see from the graph.

(B) Every constant function $f(x) = c$ is continuous. Again, this is easy to see from the graph.

(C) If two functions f, g are continuous at a point a, then their sum $f + g$ is also continuous at a. In particular, the sum of continuous functions is continuous.

(D) If two functions f, g are continuous at a point a, then their product fg is also continuous at a. In particular, the product of continuous functions is continuous.

Now, using these things conclude the followings:

I. $f(x) = x$ is continuous (this is just (A)).

II. $f(x) = x^2$ is continuous (Use (A) and (D).

III. $f(x) = x + 5$ is continuous (Use (A), (B), (C)).

IV. $f(x) = x^2 + x + 5$ is continuous (Use (B), (III), (C)).

V. $f(x) = x^3 + x^2 + 7x + 44$ is continuous.

VI. $f(x) = x^n$ is continuous (Use (D) multiple times).

VII. Any polynomial function is continuous.

Now, we add two more rules:

(E) The function $f(x) = \frac{1}{x}$ is continuous everywhere except 0.

(F) If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Now, show the followings

I. $\frac{1}{x^2}$ is continuous everywhere except 0 (You can use (E),(D) or (E), (II above), (F)).

II. $\frac{1}{x-3}$ is continuous everywhere except 3.

III. $\frac{1}{x^2-2}$ is continuous everywhere except $-\sqrt{2}, +\sqrt{2}$.

IV. For any polynomial function f, $\frac{1}{f(x)}$ is continuous everywhere except where $f(x) = 0$.

V. Any rational function $\frac{f(x)}{g(x)}$ is continuous everywhere except where $g(x) = 0$.

(3) Other than polynomial functions; all root functions ($\sqrt{x}$, $\sqrt[3]{x}$ etc), sin and cos functions, exponential functions a^x (for $a > 0$) and logarithm functions $\log_a(x)$ (for $a > 0$) are also continuous everywhere.

(4) Determine the points where the function f given by the rule

$$f(x) = \frac{x^2 - 1}{(x^2 - 2)(x^2 - 4)}$$

is not continuous.

(5) Let g be the function defined by

$$g(x) = \begin{cases} x, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

Compute the limit $\lim_{x \to 0} 2^{g(x)}$.

(6) One of the most important theorems of Calculus is the Intermediate Value Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and let K be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there is a number c in (a, b) such that $f(c) = K$.
A very nice consequence of this. Suppose that you have a continuous function f on an interval $[a, b]$. Suppose that $f(a) < 0$ and $f(b) > 0$. Then, there is a number c between a and b such that $f(c) = 0$.

(a) Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

(b) Show that the equation $x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 2 = 0$ has a root.

(7) Solve the following questions from the book:

- **(Section 2.5)** 3,4,5,10,17,18,19,20,21,22,25,36,40
- We haven't talked (and won't talk) about left and right continuity. Solve 41,42,43 and only solve their first part.
- 45,46. These are important.
- 47,51,52,53,54,55,56