Comments

See this week's extended office hours.

Product and Quotient Rules, Derivatives of Trigonometric Functions

(1) Recall from last week that, when f and g are functions differentiable at a point a, it does **not** mean that

$$(fg)'(a) = f'(a)g'(a)$$

Indeed, in our example, we saw that

$$\frac{d}{dx}x^2 = 2x \neq 1 = 1.1 = \frac{d}{dx}x\frac{d}{dx}x$$

But there is good news: *If f and g are functions differentiable at a point a, then their product is also differentiable at a.* Well, what does it mean that the product is differentiable at a? It means that the following limit exists:

$$\lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a)}{h}$$

So, let's show that this limit exists:

$$\begin{aligned}
(fg)'(a) &= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a)}{h}\\
&= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a+h)g(a)+f(a+h)g(a)-f(a)g(a)}{h}\\
&= \lim_{h\to 0} f(a+h)\frac{g(a+h)-g(a)}{h}+g(a)\frac{f(a+h)-f(a)}{h}\\
&= \lim_{h\to 0} f(a+h)\frac{g(a+h)-g(a)}{h}+\lim_{h\to 0} g(a)\frac{f(a+h)-f(a)}{h}\\
&= \lim_{h\to 0} f(a+h)\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}+\lim_{h\to 0} g(a)\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\\
&= f(a)g'(a)+g(a)f'(a)
\end{aligned}$$

So, we showed that the limit exists and we have the following rule:

$$(fg)' = f'g + fg'$$

(a) Calculate

$$\frac{d}{dx}e^x x^2$$

(b) Calculate

$$\frac{d}{dt}2^t(t^3+1)(t^4+2)$$

(2) Again, by some algebraic manipulations that we will avoid, one can show that there is also a quotient rule:

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

(a) Compute the derivative of the function f given by the rule

$$f(x) = \frac{2^x + x^2 + 1}{x^4 + 5x^2 + 4}$$

(3) Read the section 3.2.3 from Dr. Tyler Holden's notes posted on course webpage.

(4) For this course, we are not going to dive into a discussion for the following limit, but see this link

https://math.stackexchange.com/questions/75130/how-to-prove-that-lim-limits-x-to0-frac-sin-xx-1

where they give very nice explanations for why we have

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

(5) We also skip the proof for the following limit

$$\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$$

(6) Recall the following trigonometric identities from pre-calc:

$$\begin{aligned}\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y)\\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y)\end{aligned}$$

(7) Combining (5),(6) and (7), we have

$$\begin{aligned}\lim_{h\to 0} \frac{\sin(a+h) - \sin(a)}{h} &= \lim_{h\to 0} \frac{\sin(a)\cos(h) + \cos(a)\sin(h) - \sin(a)}{h}\\ &= \lim_{h\to 0} \frac{\sin(a)(\cos(h) - 1)}{h} + \frac{\cos(a)\sin(h)}{h}\\ &= \lim_{h\to 0} \sin(a)\frac{\cos(h) - 1}{h} + \lim_{h\to 0} \cos(a)\frac{\sin(h)}{h}\\ &= \sin(a)\lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(a)\lim_{h\to 0} \frac{\sin(h)}{h}\\ &= \sin(a)0 + \cos(a)1\\ &= \cos(a)\end{aligned}$$

So,

$$\frac{d}{dx}\sin(x) = \cos(x)$$

Using a similar trick, try to show that

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

(8) Read the section 3.3 from Dr. Tyler Holden's notes.

(9) Using the quotient rule and the derivatives of sin and cos, compute the derivative of tan.

(10) Compute

$$\frac{d}{dx}\frac{1}{x}$$