## Comments

This week, we are starting with derivatives. This will be our main topic until the end of the course.

## Derivatives

**(1)** There is a bust of Isaac Newton in the campus. More specifically, it is located in the MP building. Find it.

**(2)** On the coordinate plane, locate the points $(1, 1), (2, 1), (2, 1.5), (2, 2), (2, 10), (2, 100), (2, 0), (2, -1), (2, -10)$. Draw the following four line segments:

- from $(1, 1)$ to $(2, 1)$,
- from $(1, 1)$ to $(2, 1.5)$,
- from $(1, 1)$ to $(2, 2)$,
- from $(1, 1)$ to $(2, 10)$,
- from $(1, 1)$ to $(2, 100)$,
- from $(1, 1)$ to $(2, 0)$,
- from $(1, 1)$ to $(2, -1)$,
- from $(1, 1)$ to $(2, -10)$.

What can you say about these line segments? Write your observations. Can you see how the slopes of these line segments change?

**(3)** Recall that the points $(x, y)$ on the cartesian plane which satisfies the equation $y = mx + n$ (for some fixed number $m, n$) form a straight line. Refresh your middle school/high school geometry on this. $m$ is called the slope of the line $y = mx + n$. Here is an example:

- Consider the line $y = 2x + 1$. The points $(1, 3)$ and $(3, 7)$ are on this line.
- There is one and only one line between any two given points. This means that if you give me any two points on the plane, I can draw a line connecting them and I can not draw more than one straight lines connecting them.
- Going back to the first bulletpoint, we have $3 = 2.1 + 1$ and $7 = 2.3 + 1$. If you subtract the first equation from the second, you will get

$$\begin{aligned}7 - 3 &= 2.3 + 1 - (2.1 + 1)\\ &= 2.3 + 1 - 2.1 - 1\\ &= 2.3 - 2.1\\ &= 2(3 - 1)\end{aligned}$$

So,

$$\frac{7-3}{3-1} = 2$$

Well, as you can see there is nothing special about $(1,3)$ and $(3,7)$. Because we did not use their specific values. The only thing we used is the fact that they lie on the same line $y = 2x + 1$. For this reason, applying the same argument above to any two distinct points $(x_1, y_1)$and $(x_2, y_2)$ on the line $y = 2x + +1$, we see

$$\frac{y_1 - y_2}{x_1 - x_2} = 2$$

- In general, if you know that there are two points $(x_1, x_2)$ and $(x_2, y_2)$ on a line $y = mx+n$, you will always get (by applying the same argument above)

$$\frac{x_1 - x_2}{y_1 - y_2} = m$$

**(4)** We want to use these ideas to approximate rate of instantenous change of a function. We draw the graph of a function on the cartesian plane. And in Q2, we see that the bigger the change is, the bigger the slope is (in absolute value). So, there is a relation between rate of change between $f(x_1)$ and $f(x_2)$ and the slope

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

And if we want to find the **instantenous** change at a point $x_1$, we should make the point $x$ as close as to $x_1$ and keep track of the corresponding slopes:

$$\lim_{x \to x_1} \frac{f(x_1) - f(x)}{x_1 - x}$$

This is the instantenous rate of change at the point $x_1$ assuming it exists! If it exists, we say that $f$ is **differentiable** at $x_1$ and we write

$$f'(x_1) = \lim_{x \to x_1} \frac{f(x_1) - f(x)}{x_1 - x}$$

We say that $f'(x_1)$ is the **derivative** of $f$ at $x_1$. Here is a question:

1. Convince yourself that if $x$ approaches $x_1$, then $x_1 - x$ approaches zero. This is just the limit of a linear function of a variable $x$, you can do this.
2. From now on, write $h$ anywhere you see $x_1 - x$. That is, make the substitution $h = x_1 - x$. Show that this means that $x = x_1 - h$.
3. Conclude that

$$f'(x_1) = \lim_{h \to 0} \frac{f(x_1) - f(x_1 - h)}{h}$$

This is just another definition of the derivative. If you want, you can also define it as

$$f'(x_1) = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

Convince yourself that this is the same definition as above.

**(5)** As we did with limits and continuity, we will start with elementary functions. Let $c$ be a fixed real number and $f(x) = c$ be the constant function. What is the derivative of $f$ at $x = 5$?

$$f'(5) = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

Observe that we did not use the numerical value of $c$ at all. You can do the same thing with any constant function. Moreover, we did not use the numerical value of 5, either. We would have the same for $x = 2$, $x = 3$ or $x = 121314$. So, the derivative of constant function is zero at any point. And this is not so surprising, is it? Draw the graph of constant function, you will see that there is no change in the graph, so the rate of change is zero!

**(6)** How about the function $T(x) = x$? Let's find the derivative at a point $a$, this time using the other definition.

$$T'(a) = \lim_{x\to a} \frac{T(x) - T(a)}{x - a} = \lim_{x\to a} \frac{x - a}{x - a} = \lim_{x\to a} 1 = 1$$

Again, we did not use the numerical value of $a$. We always have

$$T'(a) = 1$$

**(7)** How about the absolute value function?

**(a)** Draw the graph of the absolute value function.

**(b)** Show that the absolute value function is not differentiable at 0. Hint: Compare the following right and left limits:

$$\lim_{x\to 0^-} \frac{|x| - |0|}{x - 0} \quad \text{and} \quad \lim_{x\to 0^+} \frac{|x| - |0|}{x - 0}$$

**(8)** Using limit laws and basic algebra, we can show that if $f$ and $g$ are differentiable at a point $a$, then $f + g$ is also differentiable at $a$ and $(f + g)'(a) = f'(a) + g'(a)$:

$$\begin{aligned}
(f+g)'(a) &= \lim_{h\to 0} \frac{f(a+h) + g(a+h) - f(a) - g(a)}{h} \\
&= \lim_{h\to 0} \left( \frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h} \right) \\
&= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}
\end{aligned}$$

So, when we have the sum of two functions, the situation is very nice. However, when we deal with product of two functions, we don't have a very nice situation:

**(a)** Show that if $f(x) = x^2$, then $f'(0) = 0$. Hint: Calculate

$$\lim_{x\to 0} \frac{x^2 - 0^2}{x - 0} = \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x$$

**(b)** Recall from (6) that the function $T$ defined by $T(x) = x$ has derivative 1 at every point, in particular at 0. Also, observe that $f(x) = x^2 = xx = T(x)T(x)$. However, we have $f'(0) = 0 \neq 1 = T'(0)T'(0)$.

So what is the product rule for derivatives? We will turn to this more general question later. But, for now, we will look at the derivative of polynomials.

**(c)** Recall (or show) that for any two numbers $a$ and $b$, we have

1. $a^2 - b^2 = (a - b)(a + b)$
2. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
3. $a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3)$

4. For any natural number $n$, $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$

**(d)** Using (c), show that

$$\begin{aligned}\lim_{x\to a}\frac{x^n - a^n}{x-a} &= \lim_{x\to a} x^{n-1} + x^{n-1}a + \ldots + xa^{n-2} + a^{n-1}\\ &= a^{n-1} + a^{n-2}a + \ldots + aa^{n-2} + a^{n-1}\\ &= na^{n-1}\end{aligned}$$

**(e)** Conclude that $\frac{d}{dx}x^n = nx^{n-1}$. This is a new notation for the derivative of the function $x^n$.

**(f)** Now, if $c$ is any fixed real number and $f$ is a function differentiable at $a$, we have

$$\lim_{h\to 0}\frac{cf(a+h) - cf(a)}{h} = c\lim_{h\to 0}\frac{f(a+h) - f(a)}{h} = cf'(a)$$

Hence, we can conclude $\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$.

**(g)** Using (e) and (f), compute the following derivatives

1. $f'(a)$ where $f(x) = x^2 + x + 3$,
2. $f'(a)$ where $f(x) = 5x^3 + 2x^4 + 3x$,
3. $f'(a)$ where $f(x) = a_n x^n + \ldots + a_1 x + a_0$ (any polynomial function).