Increasing - Decreasing Functions, Minima, Maxima

(1) Recall the three facts we talked about in the previous lectures:

(a) **Extreme Value Theorem.** If f is continuous on a closed interval $[a, b]$, then it attains its global minima/maxima.

(b) If f is differentiable on (a, b) and c is a local extremum for f in (a, b), then $f'(c) = 0$.

(c) If f is differentiable on (a, b) and $f'(c) = 0$ for some c in (a, b), then c is a local extremum.

Moreover, recall that if a function is defined on a closed interval $[a, b]$, one should also make sure to check the value of the function at the endpoints of the interval to determine global maximum/minimum.

(2) Recall also that if f is differentiable on (a, b) and $f'(c) > 0$ for some c in (a, b), then f is increasing near c. If $f'(c) < 0$, then f is decreasing near c. What happens if $f'(c) = 0$ for all c in the interval (a, b)? What if the derivative is always zero? Is this possible? Well, we already know that if f is a constant function, then its derivative is zero everywhere. But what about the converse? Can we say that if a function has zero derivative everywhere, then it HAS TO BE constant? Are there nonconstant functions with zero derivative everywhere? At the end of today, we are going to see that the answer is: If a function has zero derivative everywhere, then it has to be constant. But we are going to make a detour first and see an important theorem.

(3) **Mean Value Theorem.** Suppose that we have a function f which is continuous on an interval $[a, b]$ and differentiable on the open interval (a, b). Then, recall that for any number c between a and b, **the instantenous rate of change** of f at c is the derivative of c:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

and **the average rate of change** of f is

$$\frac{f(b) - f(a)}{b - a}$$

The mean value theorem says this: If f satisfies the above conditions, then there is a c between a and b such that the instantenous rate of change of f at c is actually equal to the average rate of change. That is, there is a c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

A good way to think about this is in terms of velocity. Recall that velocity is computed as the derivative of position. Suppose that two bikes are racing. Bike 1 is going with a constant speed from the beginning to the end. Suppose that these two bikes finish the race at the same time. Now answer the following two questions: a) Is it possible that Bike 2 was faster than Bike 1 from the beginning to the end? b) Is it possible that Bike 2 was slower than Bike 1 from the beginning to the end? Both answers should be no. Indeed, if Bike 2 was always faster, then Bike 2 would finish first. If Bike 2 was always slower, then Bike 2 would finish second. This tells us that these two options are not possible because we know that they tied. Hence, we conclude that at least at one point from the beginning to the end, Bike 2 had the same velocity as Bike 1. There are three options: a) Bike 2 started slower and then went faster. b) Bike 2 started faster and then went slower. c) Bike 2 had the same velocity as Bike 1 from the beginning to the end. We are not going to give a proof for this theorem. But make sure that you understand the intuition behind it.

(4) Draw a geometric picture that illustrates the Mean Value Theorem.

(5) One specific case of Mean Value Theorem is **Rolle's Theorem**. It says that if

- f is a function which is continuous on $[a, b]$
- and differentiable on (a, b)
- and also if $f(a) = f(b)$,

then there is a c between a and b such that $f'(c) = 0$. You can convince yourself easily that this is the case. Indeed, you know that $f'(c) = 0$ means that c is a local extremum. If the function starts increasing after a, it has to start decrease at some point before b. Because $f(a) = f(b)$. But the point where it starts decreasing would be a local maximum. Similarly, for the same reason, we know that if f starts decreasing after a, at some point before b it has to start increasing. The third option is that f never increases or decreases in which case it is a constant function. And the derivative is already zero everywhere.

(6) Rolle's theorem is easy to grasp. If this was a proof-based course, we would prove Rolle's Theorem first, and then we would prove Mean Value Theorem as a consequence.

(7) Let's now go back to our original question: Can we say that if a function has zero derivative everywhere, then it HAS TO BE constant? Let f be a function defined on $[a, b]$ such that $f'(c) = 0$ for every c in (a, b). **Let z be any number in** (a, b) and consider the function restricted to the interval $[a, z]$. f is still continuous on this interval and it is differentiable on (a, z). Therefore, by mean value theorem there is a $w \in (a, z)$ such that

$$f'(w) = \frac{f(z) - f(a)}{z - a}$$

But we are given that $f'(w) = 0$ in the hypothesis! This means

$$\frac{f(z) - f(a)}{z - a} = 0$$

which implies

$$f(z) - f(a) = 0$$

and hence $f(z) = f(a)$. But look at the bolded sentence above. z was arbitrary! So, for any z in (a, b) we have $f(z) = f(a)$. Therefore, the function has the same value everywhere. That is, it is constant function.

(8) Show that the function f defined by the rule

$$f(x) = \frac{x\cos(x)2^x}{x^4 + x^2 + 4}$$

has a local extremum point between 0 and 2. **Solution.** If we didn't know about Rolle's Theorem or Mean Value Theorem, we would have to compute the derivative of f and try to show that f' has a root between 0 and 2. However, now we know Rolle's theorem and can use it! Notice that $f(0) = 0$. Moreover, since $\cos(\pi/2) = 0$, we know that $f(\pi/2) = 0$. On top of this, we know that this function is continuous and differentiable everywhere! In particular, it is continuous on $[0, \pi/2]$ and differentiable on $(0, \pi/2)$. Hence, Rolle's Theorem tells us that there is a number c between 0 and $\pi/2$ such that $f'(c) = 0$. Since $\pi/2$ is less than 2, we get the result we wanted!

Note. We can't say with this method what the extremum point is. But at least we can say that there is one extremum point without computing the derivative of such a complicated function!