INCREASING - DECREASING FUNCTIONS, MINIMA, MAXIMA

(1) Recall [1] that if we have a line L (which is not parallel to y-axis) and if we have two points $(x_1, y_1), (x_2, y_2)$ on it, then the ratio

$$\frac{y_2 - y_1}{x_2 - x_1}$$

is always the same, no matter which two points on the line we pick. This ratio is called the slope of L and the line L is equal to the set of points (x, y) which satisfy the equation $y = \text{slope}\, x + n$. Here, n is another constant depending on L and can be computed by plugging in the coordinates of a point on L. We used this idea to develop the concept of derivatives during last month and we were able to compute the slope of the tangent line to a point on the graph of a given function. We are going to exploit this idea a little bit more today.

(2) We say that a function f is **strictly increasing** on an interval (a, b) if the following condition is satisfied

- If x_1, x_2 are in the interval (a, b) and $x_1 < x_2$, then $f(x_1) < f(x_2)$.

For instance the function F given by the rule $F(x) = x^2$ is strictly increasing on the interval $(2, 5)$. Indeed, if x_1, x_2 are in $(2, 5)$ and $x_1 < x_2$, we have

$$F(x_1) = x_1x_1 < x_1x_2 < x_2x_2 = F(x_2)$$

However, this function is not strictly increasing on the interval $(-5, -2)$. Explain why.

Now, let us suppose that f is a strictly increasing differentiable function on an interval (a, b). And let us take a number c in the interval (a, b). What can we say about the derivative of f at c? Recall that the derivative at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

We supposed that the function is differentiable and therefore we know that this limit exists. To say that limit exists is equivalent to say that right and left limits exist and are equal to each other. Let's investigate the right and left limits. First, let us investigate the limit from the left. So, we are interested in the case where the x-values in the following expression is less than c. But if $x < c$, then by our assumption, we have $f(x) < f(c)$. Hence, in this case we have $x - c < 0$ and also $f(x) - f(c) < 0$.

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{\text{something negative}}{\text{something negative}} = \lim_{x \to c^+} \text{something positive} > 0$$

Similarly, if we investigate the limit from the right, we have $x - c > 0$ and $f(x) - f(c) > 0$ and a similar argument will yield $f'(c) > 0$. [2]

[1] See June 13 Lecture Notes.

[2] **Important Remark:** There is a subtlety here! There is something we are sweeping under the rug. You may have a positive function like $f(x) = x$ but have a nonpositive left limit $\lim_{x \to 0^+} x = 0$. Think why this is not an issue for us.

Conclusion. If f is a strictly increasing differentiable function on an interval containing a real number c, then for any number c in this interval we have $f'(c) > 0$.

There is actually more to this conclusion. Indeed, without proof, we have the following fact: If f is a function which is differentiable on an interval containing a real number c and if $f'(c) > 0$, then f is strictly increasing on an interval containing c. For instance, the function $F(x) = x^2$ that we saw above is differentiable on the interval $(-5, 5)$. We have that $F'(x) = 2x$. Thus, $F'(2) = 4 > 0$. And indeed, on the interval $(1, 3)$ the function F is strictly increasing.

(3) Similar to what we did above, we can define strictly decreasing functions. We say that a function f is **strictly decreasing** on an interval (a, b) if the following condition is satisfied

- If x_1, x_2 are in the interval (a, b) and $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Again, similar to above we have the following two facts:

- If f is a strictly decreasing differentiable function on an interval containing a real number c, then for any number c in this interval we have $f'(c) < 0$.
- If f is a function which is differentiable on an interval containing a real number c and if $f'(c) < 0$, then f is strictly decreasing on an interval containing c.

(4) Show that the function G given by the rule $G(t) = t^3 + t$ is strictly increasing on the interval $(-\infty, \infty)$.

(5) **(a)** Give an example of an interval on which the sine function is strictly increasing.

(b) Give an example of an interval on which the sine function is strictly decreasing.

(c) Describe the behaviour of the sine function on the interval $(0, \pi)$. Compute the derivative of the sine function at $\pi/2$.

(6) Suppose that f is a function which is differentiable on an interval (a, b) and let c be in (a, b). Suppose that f is strictly increasing on the interval (a, c) and it is strictly decreasing on the interval (c, b). If we further assume that f' is continuous on (a, b), what can you say about $f'(c)$? What can you say about $f(c)$?

(7) What is the largest interval on which the function f given by the following rule is strictly decreasing?

$$f(x) = 2x^3 - 15x^2 + 36x + 17$$

(8) Suppose that f is a function with domain D and let c be in D. Then, we say that

- c is a local maximum of f if $f(x) \leq f(c)$ for all x in an interval containing c.
- c is a global/absolute maximum of f if $f(x) \leq f(c)$ for all x in D.
- c is a local minimum of f if $f(x) \geq f(c)$ for all x in an interval containing c.
- c is a global/absolute minimum of f if $f(x) \geq f(c)$ for all x in D.

(9) The following three facts are going to be our focus in the next 4 lectures:

(a) **Extreme Value Theorem.** If f is continuous on a closed interval $[a, b]$, then it attains its global minima/maxima. [3]

[3] Here the "closed" interval is essential. You can't say the same thing if you only consider your function on an open interval.

(b) If f is differentiable on (a, b) and c is a local extremum [4] for f in (a, b), then $f'(c) = 0$.

(c) If f is differentiable on (a, b) and $f'(c) = 0$ for some c in (a, b), then c is a local extremum.

(10) Find the extremum points of the function f defined on $[-2, 2]$ with the rule $f(t) = t^2$.

(11) Find the extremum points of the function f defined on $(-\infty, \infty)$ with the rule $f(t) = t^2$.

(12) Find the extremum points of the function f defined on $(-\infty, \infty)$ with the rule $f(t) = 6t^4 - 3t^2 + 2$.

(13) This topic will be our main focus on Thursday as well as next week.

[4] An extremum point is a local minimum or a local maximum.