Pigeon Hole Principle

Simple Pigeon Hole Principle

In a simple way, the Pigeon hole principle states: If there are $n+1$ pigeons in n holes then there is a hole with two pigeons in it.

Remark. Two pigeons do not mean *exactly* two. Principle means that there are *at least* two pigeons in some hole.

Really, if no hole contains two pigeons then the total number of pigeons does not exceed n.

Example. There 10 pigeons in 3 holes. Then

(a) There is a hole with 4 pigeons. Really, if no hole contains four pigeons then the total number of pigeons does not exceed 9.

(b) There is a is a hole with no more than 3 pigeons. Really, if each holes contains 4 or more pigeons then the total number of pigeons exceeds 12.

Generalization

If there are $nk+1$ pigeons in n holes then there is a hole with $k+1$ pigeons in it.

Although the principle sounds simple, it takes practice to learn which objects to consider as pigeons and which as holes.

Problems

1. Anna places 10 coins in 3 pockets of her jacket. Is it true that

   (a) There is a pocket with (at least) 4 coins?

   (b) Each pocket contains no more than 3 coins ?

   (c) one pocket (at least) contains even number of coins?

   (d) exactly one pocket contains odd number of coins ?

2. Basil keeps socks in a closet, 3 pairs of white, 2 pairs of black and 3 pairs of red, all mixed.

(a) How many socks should he take out (randomly) to be sure that there will be a pair of the same colour?

(b) The same question only there are shoes instead of socks.

3. What is the maximal number of Kings can be placed on a chessboard, so that no King is attacked?

4. There are 15 cities in a country. Each city is directly connected by a road to at least 7 other cities. The roads do not intersect except at the cities. Prove that one can reach any city from any other city by following the roads.

5. Is it possible to create a 7×7 table with each entry either of $-1, 0$ or 1, so that the sums of the numbers in each row, column and each of two diagonals are distinct?

6. There are 36 boulders weighing $495 + 24k$, $k = 0, \ldots, 35$ respectively. Is it possible to carry them out by seven 3-tons trucks?

7. On each cell of a 5×5 board sits a grasshopper. On a signal, each grasshopper jumps to an adjacent cell (by side). Prove that there is always an empty cell.

8. Prove that out of 10 positive integers, none of which is divisible by 10, one can find

   (a) two numbers whose difference is divisible by 10;

   (b) several numbers whose sum is divisible by 10.

9. (a) Do there exist four distinct positive integers, such that the sum of any three of them is a prime number?

   (b) Do there exist five distinct positive integers, such that the sum of any three of them is a prime number?

10. Rectangle 3×4 is split into 12 unit squares. Can one draw a straight line which intersects 7 unit squares (at interior points)?

11. Each of 9 given straight lines split a given square into two trapezoids with an area ratio of 2:3. Prove that at least three of the straight lines intersect at the same point.

12. There are 32 marked points on a plane, not all of which are collinear. Straight lines are drawn through all of the pairs of the points. Prove that there is a point (among marked) such that at least 7 lines intersect it.