Graphs

Graphs. Part 1

In some situations it is convenient to use graphs. Objects are represented by dots while connections between them are represented by lines or arrows. In the language of graphs, dots are called vertices while lines are called edges.

Problems

1. Some of the planets of the Solar System are connected by two way space routes; space ships travel according to following schedule:

   Mercury-Pluto, Venus-Earth, Earth-Jupiter, Mars-Earth, Jupiter-Uranus, Saturn-Neptune, Saturn-Mercury. Can one reach Mercury from Earth?

2. Line up the digits from 1 to 9 so that any two digit number made of two adjacent digits is divisible by either 7 or 13.

3. In the country Arithmetics, there are 9 cities: One, Two, Three,..., Nine. Two cities, X and Y, connected by airline is two digit number xy divisible by 3. Can one reach Nine from One?

4. (Kangaroo 2009, grade 7-8). Man Friday wrote down several different positive integers lesser than 11 in a row. Robinson Crusoe examined these numbers and noticed that, in each pair of adjacent numbers, one of the numbers is divisible by the second one. At most how many numbers did Man Friday write down?

   (A) 6 (B) 7 (C) 8 (D) 9 (E) 10

5. Is it possible to place the digits from 1 to 9 in circular way so that sum of any two adjacent numbers is not divisible by 3, 5 or 7?

6. At one 20 fstory building the lift is broken: it can only move 8 floors up or 13 floors down. Can one reach the first floor from the 20-th?

7. There is a group of seven boys. Each boy has at least three brothers among group members. Prove that all boys are brothers.

8. At a party each boy danced with 6 girls while each girl danced with 7 boys. Which group is larger, boys or girls? (Boys danced only with girls and girls danced only with boys.)

9. Each Karabas is familiar with 4 Karabases and with 7 Barabases, while each Barabas is familiar with 7 Barabases and with 9 Karabases. Which group is larger, Karabases or Barabases?

10. A set of dominos is placed in a row, according to the rules. On one end is Zero. What can be on the other end?

11. There are 20 children in a kindergarden. Any two children have the same grandfather. Prove that one of the grandfathers has at least 14 grandchildren attending the kindergarden.

12. At four corners of a 3×3 chess board there are 2 white Knights (in the bottom row) and 2 black Knights (in the top row). Is it possible to place them at the corners in alternating colours by moving the Knights according to chess rules?

13. Three chips are placed at three vertices of a regular pentagon. It is allowed to move any chip to any free vertex along a diagonal of the pentagon. Is it possible to reach the situation when one chip returns to its original position while two others trade their positions by applying this operation?

Graphs. Part 2

Definition. A vertex's degree is defined by the number of exiting edges. A vertex is called even if its degree is even and a vertex is called odd if its degree is odd.

Lemma (Handshake lemma)**.** *Total sum of vertices degrees equals the total number of edges doubled.*

(Really, any edge increases the sum of vertex degrees by exactly 2).

Corollary. *In any graph the number of odd vertices is even.*

Definition. A graph is connected if between any 2 vertices there is a path (sequence of vertices connected by edges)

Definition. A cycle is a path which starts and ends at the same vertex.

Problems

1. Each city of a country is connected with three other cities by roads. Can the total number of roads be 100?

2. Is it possible to draw on a sheet of paper 17 segments so that each segment intersects exactly 3 others?

3. In a chess tournament each of 17 participants should play with each once. It this true that in every moment there is someone who played even number of games?

4. Consider a connected graph in which every vertex is even. Prove that by removing any edge we still get a connected graph.

5. Metro Map is a connected graph. Prove that one of the stations can be closed (a vertex and all exiting edges are removed), so that one can always reach any station from any other station.

6. Bill, coming from Disneyland said that he saw 7 islands connected by bridges. From each island comes either 1, 3, or 5 bridges. Is it true that one of the islands is connected with the mainland?

7. Cities of Wonderland are connected by airlines. Each city is connected with exactly 10 of other cities except for Capital, connected with 2009 cities and Faraway connected with only a single city. Prove that one can reach Faraway city from Capital.

8. Once 11 ambassadors met at a party and exchanged handshakes. Each wanted to shake hands with the most people. By rules of etiquette, all have to exchange by the same number of handshakes. Find the maximal number of handshakes each ambassador made if the ambassadors of South Country and North Country never shake hands.

Graphs. Part 3

Definition. Traversable graph (graph that can be traced without lifting a pencil from paper or retracing an edge) is called Eulerian graph.

Theorem. *If graphs contains more than two odd vertices then it is not a Eulerian graph.*

Corollary. *Eulerian graph has either two or zero odd vertices.*

Theorem. *A graph with only two or zero odd vertices is Eulerian graph.*

Problems

1. Each of 8 vertices of a graph has degree of 2.

   Draw all such graphs. Hint: graphs are not necessary connected.

2. Can one paint the edges of a cube in white and black, so that each of two ants, First and Second can visit any vertex by moving along the edges (First ant moves along white edges , while Second ant moves along black edges).

3. There are islands, some of which are connected by bridges. A tourist visited all of the islands passing through any bridge exactly once. It happened that he had been on island Tripled three times. How many bridges lead to Tripled if:

   (a) The tourist neither started and nor finished on Tripled;

   (b) The tourist started on Tripled but did not finish on Tripled;

   (c) The tourist started and finished on Tripled;

4. Each of 11 islands is connected with each by a single bridge. A traveler wants to go along any bridge once. He started on Island of Friendship. On which island can he finish his route on?

5. Alice draws 3 circles without lifting her pencil from the paper or retracing a line. Betty draws another circle on a top of Alice's picture. It this new picture traversable?

6. In a connected graph, it is allowed to retrace each edge exactly twice. Would such a graph be traversable?

7. A country consists of several cities. Some of them are connected by Direct Express buses (each route connects two cities without intermediate stops). Mr. Poor bought one ticket for every bus route while Mr. Rich bought n tickets for every bus route (a ticket allows a single one-way travel in either direction). Both Mr. Poor and Mr. Rich started from

town A. Mr. Poor finished his travel in town B using up all his tickets without buying extra ones. Mr. Rich, after using some of his tickets, got stuck in town X: he cannot leave it without buying a new ticket. Prove that X is either A or B. (Tournament of Towns)

Graphs. Part 4

Definition. A cycle is a path which starts and ends at the same vertex.

Definition. A tree is a connected graph without cycles.

Theorem. *Any tree has a vertex of degree one. Such vertex is called Leaf.*

Theorem. *In any tree $V = E + 1$, where V and E are the number of vertices and edges respectively.*

Theorem. *Any connected graph can be transform in tree graph by removing some of its edges. This tree is called skeleton.*

Problems

1. In Wonderland there are 30 cities. Some of them are connected by roads. From any city one can reach another city by a unique route. How many roads in Wonderland?

2. Prove that a tree has at least two leaves.

3. In a graph, each vertex has a degree of 3. Prove that the graph contains a cycle.

4. There are 15 villages in a country, pairwise connected by roads. How many roads can be closed for repair, so that it is still possible to reach any village from any other village?

5. A volleyball net is a rectangle of 50×600 cells. Find the maximal number of cuts one should produce provided that the net stays in one piece.

6. On each of 11 planets an astronomer watches the closest planet (The distances between the planets are pairwise different). Prove that:

(a) There is a planet which no one watches.

(b) There are 2 planets on which astronomers watch each other.

7. Metro Map is a connected graph. Prove that one of the stations can be closed (a vertex and all exiting edges eliminated), so that from any station one can reach any other station.

8. Prove that any connected graph in which $V = E + 1$ is a tree graph.

9. Tsar Ivan the First had 3 children. 100 of his descendants had also 3 children each. The rest of them were childless. How many descendants did Tsar Ivan have?