Junior MathBattle

May 25, 2003

1. There are 8 coins on a table, arranged in a circle. All of them are identical in appearance, but three of them are counterfeits, equal in weight and heavier, than the real ones. It is known that all counterfeit coins are located one near another. What is the minimal number of weighings on a simple balance needed to identify all counterfeit coins?

2. A square is split into 100 rectangles by nine vertical and nine horizontal lines. It is known, that among rectangles there are exactly nine squares. Could it happen that there is no two congruent squares among them?

3. Two monkeys Chi and Cha found a pile of nuts (25 nuts). Chi suggested to split it in the next way: in turns they pick up the number of nuts, which is a divisor of nuts in the pile. However, it is not allowed to take the whole pile, unless it is not a single nut. Last nut is allowed to take. "I'm the First to start", Chi acclaimed. Who would get more nuts (both monkeys are smart enough)?

4. The President of Shvambrania came to the Conference (in his honor) and shook hands with all the journalists, whom he knew. All bunch of journalists knew each other, so they all exchange handshakes. On the next day one of the journalists wrote that there were 80 handshakes. Could it be true?

5. Is it possible to cut any rectangle into three pieces (by straight lines) in such a way that one can rearrange them into equilateral convex hexagon?

6. Joe keeps all his money in a Loonie Bank. He has \$500 on his account. Bank allows only two operations: a withdrawal of \$300 or a deposit of \$198. Find the maximal amount Joe can withdraw (in a few steps).

7. Peter draw a $n \times n$ grid and in each cell wrote number from 1 to n^2 (all of them, but may be in the different order). Mary draw the same grid and wanted to put the numbers in the same order that Peter did. She could choose any square (with sides on the grid) and name to Peter, then Peter would tell her all the numbers in increasing order in the corresponding square on his grid. What is the minimal number of questions Mary should ask in order to duplicate Peter's grid?

8. A number of candies were on the table. The first boy took $\frac{1}{10}$-th of them, the second boy took $\frac{1}{10}$-th of the remainder plus $\frac{1}{10}$-th of what the first boy took, the third boy $\frac{1}{10}$-th of the remainder plus $\frac{1}{10}$-th of what the first and second boys took together and so on. Which boy took the last candy? Who of these boys got more than the others?

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Junior MathBattle: Solutions

1. Answer: Two weighings is enough.

Let us numerate coins 1,2,. . . , 8.

First weighing: $\{1, 2, 3\}$ vs $\{5, 6, 7\}$.

1) $\{1, 2, 3\} = \{5, 6, 7\}$. Then either 7,8,1 or 3,4,5 are counterfeits but coins 2,6 are real for sure. Second weighing 3 vs 2 gives the final answer.
2) $\{1, 2, 3\} > \{5, 6, 7\}$ (one can consider case $\{1, 2, 3\} < \{5, 6, 7\}$ in the similar way). Then either 8,1,2 or 1,2,3 or 2,3,4 are counterfeits. Anyway, coin 2 is counterfeit for sure. Second weighing 1 vs 3 gives the final answer: if $1 > 3$ then 8,1,2 are counterfeits, if 1=3 then 1,2,3 are counterfeits, if $1 < 3$ then 2,3,4 are counterfeits.

2. Answer: No.

Proof by contradiction. Assume that there are no congruent squares. Let us consider the row and column containing one of given 9 squares. Then neither this row nor this column contains any other square (otherwise these two square would be congruent).

Removing these row and column we get new smaller square cut into 81 rectangles with exactly 8 squares among them. Repeating this procedure 9 times in the end we arrive to the 10-th square.

3. Answer: The second monkey (Cha) always wins.

1) Chi picks up 1 nut. Then Cha picks up 12 nuts and eventually wins.
2) Chi picks up 5 nuts. Then Cha picks up 1 nut, forcing Chi pick up 1 nut in his turn. Then Cha takes 9 nuts, leaving the pile of 9. At this moment Chi has 6 nuts and Cha has 10.
   - If Chi pick up 3 nuts, Cha takes 3 and wins.
   - If Chi picks up 1 nut, Cha takes 4 and wins.

4. Answer: It could be true.

Let n be the number of journalists. They exchanged by $\frac{1}{2}n(n-1)$ handshakes between themselves. We need to satisfy 2 conditions:

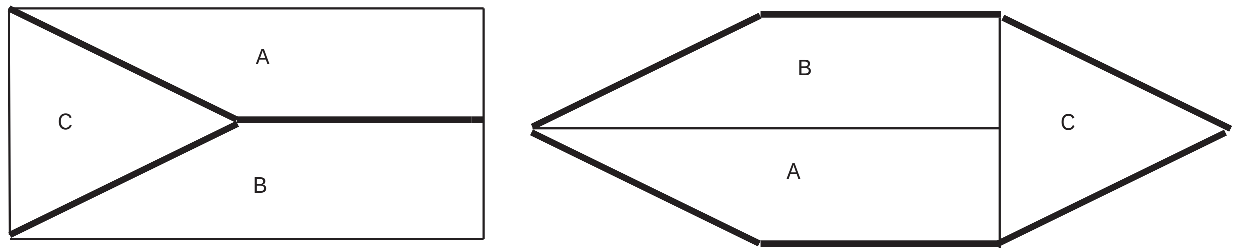
$$\begin{cases} \dfrac{1}{2}n(n-1) \le 80, \\ 80 - \dfrac{1}{2}n(n-1) \le n \end{cases}$$

which are equivalent to

$$\begin{cases} n(n-1) \le 160, \\ n(n+1) \ge 160 \end{cases}$$

which has (unique) solution $n = 13$.

5. ANSWER: Yes (see an example: all bold segments have equal lengths).



6. ANSWER: 498.

Since both 300 and 198 are multiple of 6, Joe can withdraw only multiple of 6. The largest multiple of 6 not exceeding 500 is 498. Joe can withdraw this amount:

$$500 - 300(= 200) + 198(= 398) - 300(= 98) + 198(= 296) + 198(= 494)$$

Notice that amount decreased by 6. Repeating this procedure 15 times more Joe withdraws 96 (never being below 0) and has 404. Finally,

$$404 - 300(= 104) + 198(= 302) - 300(= 2).$$

7. No solution yet

8. ANSWER: All boys took the same number of candies and therefore there were 10 boys.

Really, let n be the number of candies on the table. Then after k-th boy took his share, m candies are left on the table. So $(k + 1)$-th boy takes $\frac{1}{10}m + \frac{1}{10}(n - m) = \frac{1}{10}n$.