Junior MathBattle

May 28, 2006.

1. Could the altitude, bisector and median drawn from different vertices of a triangle create an equilateral triangle at intersection?

2. There are 300 trees in a park. If one marks any 201 trees, then there are always be an oak, a pine and a maple tree among the marked. How many oaks, pines and maple tree could be in the park (describe all possibilities)?

3. Basil places black, red and white rooks on a chessboard (the same number of each color) so that the rooks of different color do not attack each other. Find the maximal number of rooks that can be placed on the chessboard.

4. Each employee of the firm "Horns and Hoofs" is either a Lair or Truth Teller. They all are well acquainted with each other. Once, while they were sitting at a round table, every employee was asked two questions. On the first question "'How many Truth Tellers among your neighbors," only one person answered "one", all the others said "zero". On the second question if there are Lairs among his/her neighbors all answered "yes". Given this information, is it possible to distinguish the Liars from Truth Tellers?

5. Alex thinks of a number (from 1 to 9), and Boris tries to guess it. If he is mistaken, Alex changes his number: he tries to divide his number by Boris' number; however, if it does not divide evenly, he doubles it. Is there some strategy for Boris to guess Alex' number?

6. 300 oranges are distributed into 10 boxes, so that no two boxes contain the same number of fruits. A packer picks up 9 oranges from the box with the largest number of fruits and distributes them into the other boxes, one orange to each box. The packer wants to get the same number of fruits at least in two boxes. Is there a guarantee that he can ever complete the task?

Junior MathBattle. Solutions

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1. Could the altitude, bisector and median drawn from different vertices of a triangle create an equilateral triangle at intersection?

   Solution. Let CM be median, AK bisector and BH altitude of triangle ABC. Assume, that these three lines creates an equilateral triangle at intersection. Let P be intersection of CM and AK, S be intersection of CM and BH, and Q be intersection of BH and AK. Then angle OAH is $30°$ (triangle AQH is a right triangle and angle AQH is $60°$). Then angle BAK is $30°$ (AK is bisector) and angle ABH is $30°$ as well. QM is median of isosceles triangle ABQ; then angle QMB is $90°$ (since it is also the altitude of isosceles triangle). Consider triangle CHS. Angle HCS is $30°$ (angle SSH is $60°$). Then angle CMA of triangle CMA is $90°$. Given that the angle QMB is $90°$, points Q and S coincide.

2. There are 300 trees in a park. If one marks any 201 trees, then there are always be an oak, a pine and a maple tree among the marked. How many oaks, pines and maple tree could be in the park (describe all possibilities)?

   Solution. Let's say there are n_0 trees of no-name kind, n_1 of oaks, n_2 of pines, n_3 of maples. Assume that $n_1 \geq n_2 \geq n_3$. To guarantee that among 201 trees there is a maple, we need estimate $n_0 + n_1 + n_2 \leq 200$. Then estimate for maples (the least number) is $n_3 \geq 100$. This implies $n_1 = n_2 = n_3 = 100$.

3. Basil places black, red and white rooks on a chessboard (the same number of each color) so that the rooks of different color do not attack each other. Find the maximal number of rooks that can be placed on the chessboard. Solution. Lets place rooks of the first kind on squares a8, a7, b8, b7, c8, c7, the second kind on squares d6, d5, e6, e5, g6, g5, and the third kind on squares f4, f3, f2, h4, h3, h2. Rooks of different color do not attack each other. The total number of rooks is 18 and it is the maximal number given the condition. Really, no matter how one places a group of 7 rooks of one color, the sum of rows and columns that are under attack is at least 6. So, for three groups of the rooks, given the condition this number is at least $6 \times 3 = 18 > 16$, that is more than 16 (the total number of rows and columns on a chessboard). Contradiction.

4. Each employee of the firm "Horns and Hoofs" is either a Lair or Truth Teller. They all are well acquainted with each other. Once, while they were sitting at a round table, every employee was asked two questions. On the first question "'How many Truth Tellers among your neighbors," only one person answered "one", all the others said "zero". On the second question if there are Lairs among his/her neighbors all answered "yes". Given this information, is it possible to distinguish the Liars from Truth Tellers?

   Solution. Since each employee answered "'yes" on a question if there were Lairs among his/her neighbors, then for any Lair his both neighbors are Truth Tellers, and for any Truth Teller at least one of his neighbors is a Lair. Since on the question "How many Truth Tellers among your neighbors" only one answered '"one", (all the others said "zero") then no two Truth Tellers can sit next to each other (otherwise, there would be at least two answers "one"). Therefore, the person who answered "'one" is a Lair, and since Truth Tellers and Lairs alternate, it is possible to distinguish Liars from Truth Tellers.

5. Alex thinks of a number (from 1 to 9), and Boris tries to guess it. If he is mistaken, Alex changes his number: he tries to divide his number by Boris' number; however, if it does not divide evenly, he doubles it. Is there some strategy for Boris to guess Alex' number?

   SOLUTION. Strategy: Boris names the numbers in a sequence: 1, 2, 6, 4, 5, 24, 70, 16, 3 (each new term in case if the previous number was wrong). Really, if Boris names 1 and he is mistaken, then number 1 is eliminated from original list of suspected numbers (from 1 to 9). When, if Boris names 2 and he is mistaken, then number 2 is eliminated. (Really, if Alex original number is 2 then when Boris names 1, Alex divides his number (2) by 1 and gets 2 in results). In similar way naming number 6, Boris excludes original number 3 (Boris names 1, Alex divides his number (3) and gets 3; Boris names 2 and since 3 is not divisible by 2, Alex doubles his number and gets 6). In similar way, one can find the rest terms of the sequence. In worst case scenario, the answer is guaranteed on last term of the sequence.

6. 300 oranges are distributed into 10 boxes, so that no two boxes contain the same number of fruits. A packer picks up 9 oranges from the box with the largest number of fruits and distributes them into the other boxes, one orange to each box. The packer wants to get the same number of fruits at least in two boxes. Is there a guarantee that he can ever complete the task?

   SOLUTION. Let M be the maximal number of oranges , while m be the minimal number. Notice that if $M - m < 9$ then there always would be two boxes with the same number of fruits (Pigeon-Hole principle). Case $M - m = 9$ is not possible (since $1 + 2 + \cdots + 9 = 45$ and $45 + 10k \neq 300$). Assume, $M - m > 9$. Then taking out 9 fruits and distributing them into boxes according to the rule, we get boxes with $M - 1$, $i + 1$, $m + 1$ oranges. The number $m + 1$ is minimal; however, the maximal number can be at most M (if the second maximal box initially contained $M - 1$ fruit. So, the difference $M - m$ is increasing by at least one each time. Since it is finite number, then, sooner or later we reach the condition $M - m < 9$.