MathBattle

May 21, 2006.

1. Let ABC be acute triangle. Find a locus of points M such that the following conditions hold:

$$\angle MAB = \angle MCB \qquad \text{and} \qquad \angle MBA = \angle MCA.$$

Solution. It is clear that orthocenter of the triangle belongs to a locus. Inside of the triangle there is no other point with such property. Really, let M be such a point. Let A_1, B_1, and C_1 be points of intersection of AM, BM, and CM with sides BC, AC, and AB respectively. Note that quadrilateral ACA_1C_1 is cyclic. Then $\angle MA_1C_1 = \angle MCA = \angle MBC_1$, and $\angle MAC = \angle MC_1A_1$, so that quadrilateral MA_1BC_1 is also cyclic and therefore $\angle MBA_1 = \angle MC_1A_1 = \angle MAC$. Therefore, $\angle AA_1B = \angle AA_1C = \pi - \angle AA_1C$ and AA_1 is the altitude. So are BB_1 and CC_1. One can show that locus in mention also contains arc AB and points K and K'; where K (K') is intersection of diameter AO (BO) with the circle circumscribed around triangle ABC.

2. 20 students, the winners of Tournament of the Towns could not attend the award ceremony on May 28 and came one week later to pick up their book-award prizes. The books with name tags attached on them were left at front desk of Math Department. First came a crazy 6-grader. Not paying attention to the tags he chose a book and left with it. Every student who came after took his/her book if there was one, otherwise picked up any book at random. What are chances for the last student to pick up the book that was supposed for him?

Solution. Let numerate the students according their arrival; 6-grader be the first (Crazy) and John be the last. Assume that tags on books are numbered from 1 to 20, i-book is designated for i-student. Let us pair all possible outcomes into two groups; group A (consisting of outcomes where John gets his book) and group B (consisting of outcomes where John gets some other book).

Crazy is the first to come. He has three possibilities: either take its own book (we put these outcomes into group A) or take John's book (we put these outcomes into group B) or take the i-book, $i = 2, \dots, 19$. Note, that these 20 outcomes are equally likely. Now, let Crazy take i-book. Then all the students with the numbers less than i get their books, so John's book is not taken. However i-student becomes a new Crazy in the group consisting of $20 - i + 1$ students and thus the problem is reduced to the lesser number of students. By induction we conclude that the probability is always $1/2$ if number of students $N \geq 2$.

3. n points are marked on a circle. Two players in turns connect them with segments; each new segment is connected with the previous one. It is not allowed to draw the same segment twice. The player who cannot move, loses. Which of the players has the winning strategy, the first or the second? Describe the winning strategy.

Solution. First player wins. Notice that the maximal possible number of segments emitting from any vertex is $(n-1)$. Assume, that on his first move the first player connects vertices A and B; therefore, the sum of the other segments emitting from A and B is $2(n-2)$. No matter how the second player responds (he has $(n-2)$ ways) the first player draws a segment back to A. The second player is forced to draw some segment emitting from A. The first player draws a segment back to B. He keeps his moves alternating. Given that the total sum of the segments emitting from A and B is even, he is the last to enter to A (or B). After that, the second player has no move.

4. A ladder with n steps connects the ground with the roof. On each step of the ladder there is an arrow, oriented either up or down. A man moves one step up or down according to the arrow. After his move, the arrow changes its direction to the opposite one. Can it happen that the man gets stacked on the ladder forever?

   SOLUTION. Assume that the man gets stacked on the ladder forever. Then he crosses some step (say k) an infinite number of times, say, going down (then he crosses it also an infinite number of times going up). Therefore, he must cross the step $(k-1)$ also an infinite number of times since a direction of the arrow alternates. By similar arguments the man crosses the first step of the ladder an infinite number of times. However, on his second time of visiting it (if the arrow was oriented up the first time) he must step on the ground. Contradiction.

5. n cockroaches participate in the cross-corridor run. They start at the same time from a wall, run to the opposite wall of the passage, turn back and continue to run. The first cockroach moves slowly, the second one moves 2 times faster, the third one moves 2 times faster than the second and so on. Can it happen that all cockroaches meet at the same point different from the start?

   SOLUTION. Let s be width of the corridor and v be speed of the first cockroach. With no loss of the generality one can assume that $v = 1$. Let us prove that in time $t = 2s/3$ all cockroaches meet at the distance $2s/3$ (from the starting point). Let us consider the odd-numbered and even-numbered cockroaches separately (they meet running from the opposite directions).

   Let us consider $(2k+1)$-th cockroach. Then the distance that it moves in time $t = 2s/3$ is

$$D = 2^{2k} \cdot \frac{2s}{3} = 2^{2k+1} \cdot \frac{s}{3} = \frac{2s}{3} \cdot 4^k \equiv \frac{2s}{3} \mod (2s)$$

   since $4^k \equiv 1 \mod (3)$. So all odd-numbered cockroaches in time t will be at the distance $2s/3$ from the starting point.

   Let us consider $(2k)$-th cockroach. Then the distance that it moves in time $t = 2s/3$ is

$$D = 2^{2k-1} \cdot \frac{2s}{3} = 2^{2k} \cdot \frac{s}{3} = \frac{4s}{3} \cdot 4^{k-1} \equiv \frac{4s}{3} \mod (2s)$$

   thus at the distance $2s - 4s/3 = 2s/3$ from the starting point (which is exactly the point where odd-numbered cockroaches are).

   So all cockroaches meet at the same point.

6. Prove that for any n the following inequality holds

$$\sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{n}}}} < 3.$$

   SOLUTION. Let denote

$$a_k = \sqrt{k\sqrt{(k+1)\ldots\sqrt{n}}} \qquad \text{for } 2 \le k \le n.$$

Inequality $\sqrt{ab} \le (a+b)/2$ implies that

$$\begin{aligned}
a_n &= \sqrt{n} < \frac{n+1}{2},\\
a_{n-1} &= \sqrt{(n-1)\sqrt{n}} < \sqrt{(n-1)\cdot\frac{1}{2}(n+1)} < \frac{n+1}{4} + \frac{n-1}{2},\\
a_{n-2} &= \sqrt{(n-2)\sqrt{(n-1)\sqrt{n}}} = \sqrt{(n-1)a_{n-1}} < \frac{n+1}{8} + \frac{n-1}{4} + \frac{n-2}{2},\\
\dots\\
a_2 &= \sqrt{2\sqrt{3\sqrt{4\dots\sqrt{n}}}} < \frac{1}{2^{n-1}} + \frac{n}{2^{n-2}} + \frac{n-1}{2^{n-3}} + \dots + \frac{3}{4} + \frac{2}{2}.
\end{aligned}$$

By induction one can prove that

$$\frac{1}{2} + \frac{2}{4} + \dots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n};$$

therefore

$$a_2 < 3 - \frac{n+1}{2^{n-1}} < 3.$$

7. Using a compass and straightedge, construct triangle ABC given its orthocenter, midpoint of BC and the foot of the altitude dropped from B to AC.

   SOLUTION. Let H be the orthocenter, M the midpoint of BC and N the foot of the altitude dropped from B to AC.

   Using a compass and a straightedge one can construct straight line ℓ through N perpendicular to HN. Next step is to construct a circle with HM as a diameter. Let K be an intersection the circle and extension of NH. Since M the is midpoint and MK is perpendicular to HN (thus, parallel to straight line ℓ, the vertex B can be marked on the line NH so that $BK = KN$. Vertex C is marked at intersection of ℓ with extension of BM. Finally, one needs to construct a straight line through B perpendicular to HC. Then A is defined as intersection of this line with line ℓ.

8. In the top of a table, there are four deep wells, each of which contains a drinking glass. The wells are sufficiently deep and you cannot see inside of them. Each glass may be upright or inverted, but not all of them are the same way.

   Your task is to perform a sequence of moves to get each glass the same way - either all upright or all inverted. When you complete the task, a bell sounds to inform you that you have succeeded. Each move is as follows: the table rotates freely and stops at random. You may thrust your hand into at most two of the wells, feel the state of the glass and change the state of none, one or two of them, as you will. When you withdraw your hands, the table then rotates again stopping at random, and you can perform another move. Is it possible to *guarantee* after a finite number of moves that you will complete the task?

   SOLUTION Notice first that

   (a) if we somehow know the state of the system $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$ then flipping both glasses on some diagonal we solve the problem.

(b) if we somehow know the state of the system $\begin{pmatrix} + & + \\ - & - \end{pmatrix}$ then checking two adjacent glasses we can find them either in the same position or in the opposite ones. Flipping both of the glasses we solve the problem in the former case or get into situation (a) in the latter one.

(c) if we somehow know the state of the system $\begin{pmatrix} + & - \\ - & - \end{pmatrix}$ then we check two diagonal elements: in the case $\begin{pmatrix} + & \\ & - \end{pmatrix}$ we solve the problem changing $+$ to $-$; in the case of $\begin{pmatrix} - & \\ & - \end{pmatrix}$ we change one $-$ to $+$ getting case (b).

Now let us solve the problem. In the first move we just check two adjacent glasses.

On our second move we check two diagonal glasses and act accordingly:

(i) If we got $\begin{pmatrix} - & - \\ & \end{pmatrix}$ and $\begin{pmatrix} - & \\ & - \end{pmatrix}$ we know that we are in case (c).

(ii) If we got $\begin{pmatrix} - & - \\ & \end{pmatrix}$ and $\begin{pmatrix} - & \\ & + \end{pmatrix}$ we are either in cases (c) or (b); in the second move we change $+$ to $-$, getting no more than 1 $+$ and either solving the problem (bell rings), or getting into (c) for sure.

(iii) Case $\begin{pmatrix} - & - \\ & \end{pmatrix}$ and $\begin{pmatrix} + & \\ & + \end{pmatrix}$ is impossible.

(iv) If we got $\begin{pmatrix} - & + \\ & \end{pmatrix}$ and $\begin{pmatrix} - & \\ & - \end{pmatrix}$ then it is either (a) or (c) but we don't know which. Anyway, in the second move we change both $-$ to $+$. If it was (a) the problem is solved (bell rings). If it was (c) we get into (c) (with signs reversed).

(v) If we got $\begin{pmatrix} - & + \\ & \end{pmatrix}$ and $\begin{pmatrix} - & \\ & + \end{pmatrix}$ then it is either (c) or (b). Anyway, in the second move we change we change $+$ to $-$. If it was c the problem is solved (bell rings). If it was b we get into (c).

9. At one of the vertices of a cube hides a hare. Three hunters can not see it. They shoot by command (several times). Each time they hit three vertices that they choose. The hare is hit if it happens to be at one of these vertices. Otherwise, after the shot the hare runs into a neighboring vertex (connected by edge). Is there a strategy that guarantees to hit the hare?

    Solution. Denote a cube $ABCDA'B'C'D'$. Example of strategy: $A'C'D$; $A'C'D$; $A'C'D$; $AB'C$; $AB'C$.