MathBattle

February 6, 2006.

1. A circle with a fixed diameter and a fixed point on its circumference is given. Using a straightedge, construct a perpendicular to the given diameter through the given point.

   Solution. First, let us consider a case when the fixed point (X) is inside of the circle. Let A and B be ends of the given diameter. Connect A and B with X and extend straight lines until they intersect the circle; let points of intersection be$M$ and$N$ respectively. Let D be point of intersection of AN and BM. In triangle ADB, AM and BN are altitudes, so is the extention of DX.

   Let us consider the case when the fixed point (X) is on the circle. First, let us pick up any point inside of the circle and construct a perpendicular to the given diameter through it. Let $Y$and Y' be the points of intersection this perpendicular and the circle (Y being on the same side as X). Let O be a point of intersection of XY' and the diameter. Let X' be a point of intersection of extension of YO and the circle. Then, XX' is the perpendicular in question.

2. Each point of 3D space is coloured either white or black. Is it (always) true that there exists an equilateral triangle with side length 1 meter such that all its vertices are of the same colour?

   Answer Yes, it is true.

   Solution Assume, on the contrary, that no such triangle exists. Consider a regular tetragon $ABCS$ with edge 1. The only way to color its vertices providing the assumption is to color, say, A, B, in one color (black) while C and S in another color (white). Consider tetragon $ABCS'$ symmetrical to $ABCS$ with respect to plane ABC. Notice, that vertices S and S' have the same color. Therefore, all the points of 3D space within the distance $SS' = 2\sqrt{2/3}$ have the same color (otherwise the question is proved). Consider a sphere with center at S and radius of SS'. Since all points of this sphere have the same color and the radius $SS' > 1$ then there is a triangle on the sphere satisfying the conditions.

3. A grasshopper can jump a distance 1 in the directions parallel to the sides of a certain regular heptagon, each time choosing one of 14 possible vectors. Somewhere on the plane there is a seed can of radius 0.01. Is it true that the grasshopper can always reach the seed can?

   Answer Yes.

   Solution Let us construct a regular heptagon $M = ABCDEFG$ with sides of the length 1 parallel to the sides of the original heptagon. Then Grasshopper can jump in the direction parallel to any side of M a distance equal the side.

   Consider $\vec{AB}$, $\vec{BC}$,..., $\vec{GA}$. Drawing these vectors from some point Q_1 we get a regular heptagon $M_1 = A_1B_1C_1D_1E_1F_1G_1$ with side $k = 2\sin(\pi/7) < 1$. We claim that in two jumps Grasshopper can jump in the direction parallel to any side of M_1 a distance equal the side. Really, Grasshopper can move from A_1 to B_1 in two jumps (from A_1 to Q_1 and from Q_1 to B_1)which are of the length 1 and parallel to sides of M.

   Repeating this procedure we can construct a regular heptagon M_2 with side k^2, a regular heptagon M_3 with side k^3 and so on, a regular heptagon M_n with the side $\epsilon = k^n$, where ϵ could be arbitrary small. Therefore, in finite number of jumps Grasshopper can move in the direction parallel to any side of M_n on the distance ϵ.

   Note, that from an original position O one can always reach the center R of the seed can, moving parallel to two sides of M_n on the distances x and y respectively. However, Grasshopper can jump on the distances x' and y' multiples of ϵ, (different from x and y by no more than $\epsilon/2$); thus, slightly missing the center but still getting into the seed can.

4. An ant crawls on the surface of a cube going from vertex to vertex either by edge or by face diagonal. It is not allowed to intersect the path or to visit the same vertex twice. Find the maximal length of the path from one vertex to the opposite vertex of the cube.

   ANSWER $3+4\sqrt{2}$.

   SOLUTION Assume that side of a cube is 1 unit. Since cube has 8 vertices, the path in mention consists of no more than 7 segments. Its total length is a+b sqrt(2) . Let us color the vertices of the cube alternately into 2 colours, black and white. Notice, that starting vertex and ending vertex have different colours; therefore, going along the path the colour is changing the odd number of times. Also notice that going along the edge the colour is changing while going along the diagonal is not. Therefore, a is an odd number. Let us consider a case $a = 1$. Since the length of the path is $(1+6\sqrt{2})$, the path starts or ends with at least three diagonals in row. It is clear that case of 4 diagonals in row is impossible provided the conditions. Therefore, the only way for the path to exist, is: three diagonals, an edge, three diagonals. One can draw paths consisting of three diagonals in row from 2 opposite vertices; however, these paths are not connected by edge. Example of the path $3+4\sqrt{2}$: $A - A_1 - D - B - B_1 - C - D_1 - C_1$.

5. In Dracula's dungeon there are n^2 caves, connected by tunnels (square grid with n^2 knots). Two young vampires play the following game. In alternating turns, they destroy tunnels by filling them with soil. In each move one vampire destroys one or two tunnels so that all the caves are still connected. Vampire who cannot make a move loses. Which of the vampires has a winning strategy, the first or the second?

   ANSWER If $n = 3k+1$, then Second vampire wins, otherwise First vampire wins.

   SOLUTION Let us consider a graph in which vertices represent caves while edges represent tunnels. Simple connected graph (tree) with n^2 vertices has n^2-1 edges. Since the number of the edges of the given graph is $2n(n-1)$ then the number of edges to be removed preserving the connectivity is $2n(n-1)-(n^2-1) = n^2-2n+1 = (n-1)^2$. Since each vampire destroys one or two tunnels at the time, it is easy to see that if the number of tunnels is multiple of 3 then Second vampires wins; otherwise, First vampire wins.

6. A regular hexagon with side length 1 is split into equilateral triangles. Find the total sum of areas of the circles inscribed in these triangles.

   ANSWER $\pi/2$

   SOLUTION Notice that in case of equilateral triangle the ratio of area of (inscribed) circle to area of the triangle is constant (c= $\pi/3\sqrt{3}$). Therefore, for any partition of hexagon into equilateral triangles the total sum of areas of the circles inscribed in these triangles equals cA, or $6c\sqrt{3}/4 = \pi/2$, where A is area of the hexagon.

7. In a dark room 20×20 m^2 John is chasing a cockroach. Jonh's flashlight illuminates a circle of radius 2 m, Jonh's speed is 2m/sec and the cockroach's speed is 0.2m/sec. Is there a strategy for John to catch the cockroach (the initial position of cockroach is unknown)?

   ANSWER

   SOLUTION Note, that if John sees the cockroach, he catches it.

   Let us describe a strategy that guarantees to detect the cockroach. Assume that walls of the room are oriented in $\mathcal{N}-S$ and $\mathcal{E}-W$ direction. Let John chase the cockroach to east, not giving it the opportunity to go west undetected. Let John go from A to B in $\mathcal{N}$ direction, then (when the distance to the wall is x) turn to $\mathcal{E}$ going from B to C a distance a, then turn to $\mathcal{S}$ and go from C to D(until the distance to the wall is x) .

Note that to cover the path from A to D) with total length of $(20-2x)2+a$, John needs to spend $(20-2x)+a/2$ sec. and that the width of illuminated stripe is $2\sqrt{r^2-x^2}-a$, where r is the radius of flashlight. If time for cockroach to cover the distance $2\sqrt{r^2-x^2}-a$ is bigger than John, then cockroach will be detected during John by-pass.

So the question reduced to solving inequality $(20-2x)+a/2 < (2\sqrt{r^2-x^2}-a\ )/\ 0.2$. One may check that, for instance a pair $x=1/2$ and $a=0.05$ is a solution.

8. Nine weights, of masses 1 to 9 grams, are arranged in clockwise ascending order in a circle. All weights look the same. Using a simple balance identify the 1 gram weight in the minimal number of weightings (the simple balance show if both side are in equilibrium, or, if they unequal, which is heavier).

   Answer 2 weighings.

   Solution Let denote the weights as A, B, C, D, E, F, G, H, I in the clockwise direction.

   First weighing: $A+C$ **?** $F+G$

   One may check that

   i) if $A+C>F+G$ then weights D, E, F are under suspicion,
   ii) if $A+C<F+G$ then weights A, H, I are under suspicion,
   iii) if $A+C=F+G$ then weights B, C, G are under suspicion.

   Second weighting in case i) $C+F$ **?** $D+E$

   - if $C+F>D+E$ then $D=1$,
   - if $C+F=D+E$ then $E=1$,
   - if $C+F<D+E$ then $F=1$.

   Second weighting in case ii) $G+A$ **?** $H+I$ is considered in the similar way.

   Second weighting in case iii) $B+G$ **?** $C+F$

   - if $B+G>C+F$ then $C=1$,
   - if $B+G=C+F$ then $B=1$,
   - if $B+G<C+F$ then $G=1$.

9. Find all values of x such that $\tan(x)$ and $\tan(2x)$ are both integers.

   Answer $x=\pi k$

   Solution Let $\tan(x)=n$, $\tan(2x)=m$. Then $m=\dfrac{2n}{(1-n^2)}$ or $-mn=2+\dfrac{2}{(n^2-1)}$. Therefore, the number $A=\dfrac{2}{(n^2-1)}$ is integer. So, if $n=0$ then $A=-2$, if $|n|=1$ then A does not exist, if $|n|>1$ then $0<A<1$.

10. 5 solders, A, B, C, D, and E in turns (but not necessarily in this order) are working on the disposal of an explosive device. Each of them made one attempt. A spent twice more time than B, B twice more than C, while D and E spent the amount of time as C. Every 10 min the solder on duty reported the state of the work done. Could it happen that each of them reported exactly once?

    Answer No.

Solution Assume, that each of the solders reported exactly once. Then A worked less than 4 min, B less than 2 min, while C, D and E less than 1 min. Therefore, no two of C, D and E had been working in row. Then the order in which they worked was the first, the third and the fifth, while A and B were the second and the fourth. Then B together with the solder before him and the solder after him spent less than 4 min; therefore, each of them could not report exactly once.

11. In triangle ABC AE is bisector and BH is altitude. $\angle A$ is acute and $\angle AEB = 45°$. Find $\angle EHC$.

    Answer 45°.

    Solution Let B' be a point symmetrical to B with respect to bisector AE. Then B' is on AC,(between C and H) and BE=EB'. Therefore, $\angle BEB' = 2\angle AEB = 90°$ and $\angle EBB' = \angle EB'B$ =45°. Note, that quadrilateral $BEB'H$ is cyclic, thus $\angle EBB' = \angle EHB'$. Then $\angle EHC = 45°$.