MathBattle

April 17, 2005.

Problems offered for all Leagues

1. Three cops are trying to catch a gangster on an infinite square grid. All four move along the horizontal and vertical lines of the grid at the same maximal speed. The gangster is caught if he is ever on the same line as a cop. At the start time the location of the gangster is unknown. Is there a strategy that ensures that the cops catch the gangster eventually?

   SOLUTION (i) Let the length of the segments of the grid be 1 and the speed be 1. Let us assume that the three cops, (A, B, C), are gathered at point O, the origin of the coordinate system, and the gangster (G) is at a distance of no more than r (r is integer) from O.

   Cops Strategy: A runs to the West and B runs to the East while C patrols along OY between $(0, -2r)$ and $(0, 2r)$ resting for 1 minute at each intersection.

   Let us show that after r passes C, the x- coordinate of G is between those of A and B. If G even once went to outside of the y-interval $[-2r, 2r]$, then he made at least r vertical moves and thus the x-coordinates of both A and B exceeded (or reached) the x-coordinate of G. On the other hand, if the y-coordinate of G never exceeded $2r$, then during each pass of C he must at least once go vertically because if he stayed on the same horizontal line, he would be caught by C.

   After we have ensured that the x-coordinate of G is been between those of A and B, let us change the strategy. Now A runs to the North, B run to the South, while C patrols in an East-West direction between the verticals of A and B, staying at each knot for one minute. Note that G cannot cross these verticals. Further, after some time the y-coordinate of G is localized between those of A and B. Therefore, G is somewhere inside of the rectangle with A and B be as opposite vertices.

   Further strategy of catching G is obvious (if he had not been caught so far). We denote this strategy by $\Sigma(r)$.

   (ii) Assume that G is at a distance of R from the origin (but the cops do not know R). They employ the strategy $\Sigma(r)$.

   If G is not caught, then our initial assumption is wrong. Then cops return to point O. Note that the time $t(r)$ that the cops spent trying to catch G according to $\Sigma(r)$ (and return to O) depends on r only. During this time G will be at distance $R + t(r)$ from O.

   Now, let us apply the strategy $\Sigma(r_1)$ with $r_1 = 2r + t(r)$; then strategy $\Sigma(r_2)$ with $r_2 = 3r + t(r) + t(r_1)$ and so on. G will be be caught eventually, since if he is not caught after $\Sigma(r_k)$ it would mean that at the initial time he was at a distance of $R \geq kr$ from O.

2. A positive non-straight angle with vertex O and a point P inside of it are given. Construct a segment through P creating a triangle of minimal area.

   SOLUTION. Point P must be the midpoint of the segment in question. To see this, let us consider any other segment through P and the triangle created by it. Its area is equal to the area of the triangle containing the segment in question plus and minus the areas of the two triangles created by both segments. One can prove that the triangle to be added has a larger area than the triangle to be subtracted.

CONSTRUCTION: Through point O' symmetrical to O with respect to P draw two lines parallel to the sides of the angle. Then P is the centre of the created parallelogram.

3. Each inhabitant of the Three Parties Island is either a Truth- Teller or a Liar and a member of exactly one of the three political parties. One day each person was asked whether he belonged to

   (a) the First Party

   (b) the Second Party

   (c) the Third Party

   60%, 50%, and 40% of population answered affirmatively on questions (a), (b), and (c), respectively. Who (Truth-Tellers or Liars) constitutes a majority of the Second Party?

   SOLUTION

   Each Truthteller answered affirmatively to exactly one question while each Liar did so on two questions. Note that the total number of positive answers is 150=40+50+60% (of the population). Therefore, the Liars constitute exactly 50% of the population.

   The second question was answered positively by either Truthtellers from The Second Party or by Liars who are not from The Second Party. Let Truthtellers from The Second Party constitute $x\%$ of the total population; then Liars who are not from The Second Party constitute $(50 - x)\%$ of the population. Also, the number of Liars from The Second Party constitute $50\% - (50 - x)\% = x\%$ of the population. Therefore, The Second Party is equally split between Liars and Truthtellers.

4. Find a minimal value for x, where x is positive and $\lfloor x \rfloor^2 - x\lfloor x \rfloor + 3 \leq 0$?

   ANSWER: $4\frac{3}{4}$.

   SOLUTION. Substituting $x = \lfloor x \rfloor + \{x\}$ into the inequality we get $\lfloor x \rfloor \cdot \{x\} \geq 3$. Since $\{x\} < 1$ then $\lfloor x \rfloor > 3$. If $\lfloor x \rfloor = 4$ then $\{x\} \geq 3\lfloor x \rfloor = 3/4$, therefore $x \geq 4\frac{3}{4}$. If $\lfloor x \rfloor \geq 5$ then $x\lfloor x \rfloor = 5 > 4\frac{3}{4}$. On the other hand, for $x = 4\frac{3}{4}$ $\lfloor x \rfloor = 4$ and $\lfloor x \rfloor^2 - x\lfloor x \rfloor + 3 = 0$.

5. Let $f(x) = x^2 + ax + b$, where a and b are integers, $|f(0)| \leq 800$ and $f(120)$ is a prime number. Could $f(x)$ have an integer root?

   SOLUTION. Assume that $f(x)$ has an integer root x_1. Then $f(x)$ has another integer root $x_2 = -a - x_1$. Therefore the prime number $f(120) = (120 - x_1)(120 - x_2)$. It means that $(120 - x_1) = \pm 1$ while $(120 - x_2)$ is prime (or vice versa). Therefore, $x_1 \geq 119$ while $|x_2| \geq 7$ (since 113 and 127 the closest primes to 120. Then $|f(0)| = |x_1 x_2| \geq 7 \cdot 119 > 800$. Contradiction. The answer to the question is negative.

6. What is the minimal width of a long strip that contains every triangle of unit area when suitably rotated?

   SOLUTION. Let us note that the side of the equilateral triangle of area 1 is $2/\sqrt[4]{3}$ and its altitude is $\sqrt[4]{3}$).

   One can prove that by rotation of that triangle one cannot decrease the width of the stripe. Now, let us prove that every triangle of area 1 has an altitude not exceeding $\sqrt[4]{3}$. It is enough to prove that a triangle has a side which is no less than $2\sqrt[4]{3}$. Assume, that every side of the triangle is less than $2/\sqrt[4]{3}$. Let α be the minimal angle of the triangle. Note, that α does not exceed $60°$. Then the area of the triangle is less than 1, contradicting the hypothesis.

Problems offered for League II only

7. A man is on a straight road, which passes through fields. He can go along the road at 6 km/h or through the fields at 2 km/h. Find the locus of points that he can reach in one hour.

SOLUTION. Consider the part of the set in the right half-plane. The fastest way to reach each point (one can prove it) is to walk along the road and then to walk along a straight line through a field. Therefore the set in question is the union of disks $D(6t, 2(1-t))$ of radii $2(1-t)$ with the centers at $(6t, 0)$, $0 \leq t \leq 1$. If we drop the condition $t \geq 0$ we will get a sector bounded by straight rays. However the condition $t \geq 0$ reduces this domain and the answer will be the domain bounded by two arcs of the circumference $C(0, 2)$ and by four straight segments passing through points $(\pm 6, 0)$ and tangent to $C(0, 2)$.

8. Among 100 coins of value 1, 2, 3, ..., 100 tugriks there are no more than 20 counterfeits. A coin is counterfeit if its weight in grams is not equal to its value. Using a simple balance and fewer than 50 weighings determine whether the 10-tugrik coin is a genuine one.

SOLUTION. Let us check the following 49 equalities:

$$\begin{array}{ccccc} & 20+10=30 & 40+10=50 & 60+10=70 & 80+10=90 \\ 1+10=11 & 21+10=31 & 41+10=51 & 61+10=71 & 81+10=91 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 9+10+19 & 29+10=39 & 49+10=59 & 69+10=79 & 89+10=99 \end{array}$$

where numbers denote nominal values of the coins.

Note that every coin except 10 and 100 tugriks has been used exactly once. If the 10-tugrik coin is genuine, then no more than 20 equilibriums would be violated.

On the other hand, if the 10-tugrik coin is counterfeit then at least 30 equilibriums (consisting of two genuine coins) would be violated.

Problems offered for League I only

10. Two players play the following game. The First player thinks of a number and the Second player tries to learn it by naming any k positive integers each not exceeding 100. The First player responds by telling the sum of his number and one of the numbers named by the Second player. Determine the maximal value of k such that Second player can ensure that he eventually learns the number.

ANSWER 50.

SOLUTION. Let us prove that if $k \leq 50$ then the Second player can learn the number. Let B be an unknown number and a_i $(i = 1, \ldots, k)$ be the first series of numbers called by the Second Player.

after naming first series the Second player knows k the numbers which are "under suspicion"; that is, $B_i = (A - a_i), i \leq k$, where A is the response of First Player.

Let $d = B_1 - B_2$. In order to exclude either B_1 or B_2, the Second player could name k numbers, each not exceeding 100, with no two differing by d.

For example, such a sequence could be

$$\begin{array}{cccc} 1, & 2, & \dots & d, \\ (2d+1), & (2d+2), & \dots & 3d, \\ (4d+1), & (4d+2), & \dots & 5d, \\ \vdots & \vdots & \ddots & \vdots \\ (2md+1), & (2md+2), & \dots & (2m+1)d \end{array}$$

One can show that there are no fewer than 50 numbers in this sequence.

On the other hand, among 51 positive integers not exceeding 100 there are always two consecutive numbers and therefore the First Player can always answer in such a way that the Second Player cannot choose between the numbers A and $A+1$.

11. Problem 11 with $u = 6$.

Problems offered for League I only

11. A man is at the intersection point of two straight roads that form a right angle, which are surrounded by fields. He can go along the one road at U km/h, or along another road at u km/h or through the fields at v km/h ($v < u \le U$) . Find the locus of points that he can reach in one hour.

    Solution. Consider the part of the set in the right half-plane. The fastest way to reach any given point (one can prove it) is to walk along one of the roads and then to walk along a straight line through a field.

    (i) Consider first the case of the road along axis OX. Then the set in question is the union of disks $D(Ut, v(1-t))$ of radii $v(1-t)$ with centres at $(Ut, 0)$, $0 \le t \le 1$. If we drop the condition $t \ge 0$ we will get a sector bounded by straight rays. However condition $t \ge 0$ reduces this domain and the answer will be the domain bounded by two arcs of the circumference $C(0, v)$ and by four straight segments passing through points $(\pm U, 0)$ and $(0, \pm u)$ and tangent to $C(0, v)$.

    (ii) Let $v_0 = uU/\sqrt{U^2 + u^2}$. In the case of two roads along axes OX and OY we get the union of the original domain $G(U, v)$ and the similar domain $G(u, v)$ rotated by 90°. If $v < v_0$ we shall have a star-like figure bounded by four rays. If $v = v_0$ we shall have a rhombus with vertices $(\pm U, 0)$ and $(0, \pm u)$ and if $v_0 < v < u$ our figure will be bounded by 4 arcs and 8 straight segments.

12. A magician and two of his assistants perform the following trick. A Volunteer from the audience gets a deck of cards with the numbers from 1 to $2n + 1$ on them, $n \ge 6$. He chooses one card and keeps it, while splitting the rest of the deck equally between the assistants. Without contacting each other, each one chooses two cards from his deck. Then they put them together into a pile and submit it to Magician. The Magician inspects the cards and names the one the Volunteer chose. Explain how this trick works.

SOLUTION. Each Assistant should communicate to the Magician the sum s ($s = 0, \ldots, 2n$) of the numbers of his pile modulo $(2n+1)$ (then Magician could name the card). Prior to the trick the trio can agree that the order of the pairs of cards (ascending or descending) describes the Assistants' combination of cards. Here is an example of how it could be done.

(a) If an Assistant has either a pair $\{s-k, s+k\}$ for some $k = 2, \ldots, n$ or $\{s, s+2\}$ then he shows this pair in the ascending order. Otherwise he shows cards in descending order.

(b) Namely, if Assistant has either a pair $\{s+1, s-5\}$, or $\{s+5, s+1\}$, or $\{s-1, s-4\}$, or $\{s+4, s-1\}$, $\{s, s-2\}$ he shows this pair in the descending order (note that the difference of the cards in each case is distinct).

To justify this strategy we need to prove that there is always a pair of one of the indicated types. Assume that this is not the case. Let us split the set $\{1, 2, \ldots, 2n+1\}$ into the following three triples $\{s+1, s+5, s-5\}$, $\{s-1, s+4, s-4\}$, $\{s, s+2, s-2\}$, one pair $\{s-3, s+3\}$ and $(n-5)$ pairs $\{s-k, s+k\}$ with $k = 6, \ldots, n$. Due to our assumption Assistant has no more than 1 card from each set and therefore he has no more than $(n-1)$ cards, contrary to hypothesis.