# MathBattle

**December 12, 2004.**

1. Using only a compass divide a given square into two parts of equal areas.

   Solution. Let $ABCD$ be a square given. Assume that we know its center $O$ (somehow). Then construct part of circumference with the center $A$ and radius $AO$ from $O$ to an intersection with the side $AB$ and another part consisting of circumference with the center $C$ and radius $CO$ ($AO$) from $O$ to an intersection with the side $CD$. It is easy to see that this line divide the square into two parts of equal areas.

   Now let us show how to construct the center of the given segment ($AC$) using only a compass. First let us construct a point $A'$ symmetrical to $A$ with respect to $C$. For that construct a circumference with the center $A$ and radius $AC$. Then lay aside on circumference three times the segment $AC$, starting from $A$ . Last point is $A'$.

   Now construct two circles, the first with the center $A$ and radius $AC$ and the second with the center $A'$ and radius $AA'$. Let $F$ and $E$ be points of their intersection. Construct two new circles with the centers $F$ and $E$ and radius $AC$. They intersect at point $O$ (the middle of $AC$). Proof follows from similarity of two isosceles triangles $AA'F$ and $AFO$.

2. Triangle $ABC$ with angle $A$ acute is inscribed into a circle. Find the value of angle $B$, if it is known that the diameter of the circle passing through point $H$ ($BH$ is a hight of $\triangle ABC$) splits the triangle into two parts of equal areas.

   Solution. Let $O$ be the second point of intersection the diameter in mention with $\triangle ABC$. Let $M$ be a midpoint of side $AC$. Then $BM$ divides $ABC$ into triangles of equal areas.

   (a) $BM$ belongs to diameter in mention. Then $M = H$, $\triangle ABC$ is isosceles and $\angle B = \pi - 2\angle A$.

   (b) Let $K$ be a point of intersection of $BM$ and the diameter in mention. Then triangles $BOK$ and $HKM$ have equal areas, so do triangles $BOH$ and $BMH$. Then $BH \parallel OM$ and therefore $OM$ is midpoint perpendicular to $AC$ meaning that $OM$ belongs to some diameter. Then point $O$ is the center of the circle since it belongs two different diameters. If $O$ belongs to $BC$, then $A$ is the right angle, that contradicts to the condition. If $O$ belongs to $AB$, then $C$ is the right angle, and $\angle B = \pi/2 - \angle A$.

3. Among $n$ knights each two are either friends or foes. Each knight has exactly three foes and all foes of his friends are also his foes. Find all possible values of $n$.

   Solution. Since each knight has exactly three foes then the total number of all pairs of foes equals to $3n/2$. Therefore $n$ is even number. Let us estimate $n$ the number of knights. It is clear that $n \geq 4$. Let us show that $n < 7$. Let $A$ and $B$ be two foes and let us add any five the other knights to them (so, we would have a group of seven). Then among these five $A$ has no more than 2 foes, so at least three are his friends. Then these three have to be foes of $B$, meaning that $B$ has at least 4 foes (one is $A$) . Contradiction. One could construct examples showing that either $n = 4$ or $n = 6$ is possible.

4. Each of 2004 boxes contains one pebble. Some of pebbles are white and the total number of them is even. It is allowed to chose any two boxes and ask questions whether there is a white

ball (at least one) among of chosen two. Find the minimal number of questions that allow to determine two boxes which contain white pebbles.

Solution. Answer: 4005.

Let us numerate boxes (and pebbles in them) by the numbers from 1 to 2004. Let pair $(i, j)$ denote a question. Show, that 4005 questions is enough. Really, consider the following sequence of the questions: $(1, 2), (1, 3), \ldots, (1, 2004), (2, 3), (2, 4), \ldots, (2, 2004)$. If all the answers are positive then first and second pebbles are white. (If first pebble is, say, black, then pebbles from 2 to 2004 are white and the total number of white pebbles is odd. Contradiction). If one of the answers is negative (say on $(1, k)$ question), then pebble number 1 is black. Therefore, pebble $j$ is white if on questions (1, j) the answer was positive. Let us prove, that 4005 is the minimal number. Assume, that there is a procedure which allows to determine two white pebbles for the lesser number of questions. Let to answer positively on all the questions. Then after 4004 questions (as maximum) one is able to determine two boxes, say m and n with white pebbles. Note, that one of the questions either (m,i) or (n,i) is unanswered (since the total number of such questions is 4005 ). Let it be (m, k) . Let us place into m and k boxes black pebbles. Then all the answers stay unchanged: however, box m contains black pebble.

5. A monkey has two coconuts. It is fooling around by throwing coconut down from the balconies of $M$-storey building. The monkey wants to know the lowest floor when coconut is broken. What is the minimal number of attempts needed to establish that fact, if

   (a) $M = 15$.

   (b) $M$ is arbitrary.

Solution. Let us note that if the monkey has one coconut and the building is $M = k$ storey high, then the number of attempts needed equals $k$. Also, in case of $k$ attempts permitted, the maximal high of the building is $k$.

Consider the case of two coconuts. Let us fix number $k$, attempts permitted and find the maximal hight of the building when the answer on experiment is guaranteed. With two coconuts monkey starts from the $k$-th floor (if coconut is broken, it still has $k-1$ attempts to investigate floors from the first to $k-1$ to get the answer). If coconut is not broken, then monkey spends it second attempt, going up $k-1$ floors (it is on $(2k-1)$-th floor now) to continue its experiment. Again, if coconut is broken, then $k-2$ attempts is enough to answer the question, otherwise the monkey goes up $k-2$ floors. So, the maximal high of the building in case of $k$ attempts permitted is $k + (k-1) + (k-2) + \cdots + 1 = k(k-1)/2$. So, in order to evaluate the minimal numbers of attempts given the $M$ is the high of the building, we should find the minimal $k$ which satisfies inequality $k(k-1)/2 \geq M$

6. There are $N$ trenches in a row. A soldier is hiding in one of them. A gunman shots at any particular trench, and kills the soldier if he is there. After each shot the soldier (if alive) runs to an adjacent trench. Is there a strategy for the gunman that ensures the kill after the number of shots?

   (a) $N = 2004$.

   (b) $N = isarbitrary.$

Answer: yes.

Solution. (i) Consider the case when $N = 2k$ is even. Strategy: Gunman shots in sequence from number $1, 2, \ldots, 2k, 2k-1, 2k-2, \ldots, 1$. Let us assume, that the soldier is hiding in a trench with odd number. Note, that after any odd number of shots the soldier be in even numbered trench, while after even number of shots he be in odd numbered trench.

Let us verify, that the above strategy works. Gunman shots at trench numbered 1. Let us assume, that the soldier was not there, otherwise he is killed. After the first shot the soldier would be in an even numbered trench. The gunman shots at trench numbered 2. Assuming, that soldier was not there, after the shot he can occupy any odd numbered trench, except number 1. The gunman shots at trench numbered 3. Assuming, that soldier was not there, after the shot he can be in any even numbered trench, except 2. The gunman shots at trench numbered 4. Assuming that soldier was not there, after the shot he can be in any odd numbered trench, except number 3 and 1 (to run to number 3 he needs to be in trenches 4 or 2 prior to it, which is impossible). And so on. To a moment the gunman shots at 2k trench, he kills the soldier for sure unless our assumption was wrong and the soldier was hiding initially in an even numbered trench. So, after the gunman shot, we learn, that the soldier was not there, so prior to that he could occupy an any even numbered trench, except $2k$. Next shot persuades us he is not in $(2k-1)$ trench, but in another trench odd numbered. The shot at $(2k-2)$ persuade us he was not in $(2k-3)$ prior to it. . . And so on.

7. 11 coins identical in appearance are arranged in a row. Among them there are 3 counterfeits, with the same weight, heavier than the real ones, and all three of them are placed next to each other. Using a simple balance find the minimal number of weighings that allow to determine all counterfeit coins. The simple balance shows whether both sides of the balance are at equilibrium or one side is heavier/lighter.

   Solution.

   Show that it could be done in two weighings.

   Let us numerate the coins from 1 to 11. Let us start from weighing

   **1**. (1,2,3) ? (9,10,11).

   (a) In case of equality $(1, 2, 3) = (9, 10, 11)$ counterfeits are among (4,5,6,7,8) with 6 be counterfeit for sure. Then

      **2**. 5 ? 7. In case of 5=7 counterfeits are (5,6,7); if $5 > 7$ then 7 is a real one and counterfeits are (4,5,6). If $5 < 7$ then counterfeits are (6,7,8).

   (b) If $(1, 2, 3) > (9, 10, 11)$ then counterfeits are among (1,2,3,4,5) with 3 be counterfeit for sure. Then

      **2**. 2 ? 4. In case of 2=4 counterfeits are (2,3,4); if $2 > 4$ then 4 is a real one and counterfeits are (1,2,3). If $2 < 4$ then counterfeits are (3,4,5).

   (c) $(1, 2, 3) < (9, 10, 11)$ is considered in the similar way.

   One weighing in not enough to determine counterfeit coins.

8. Circle is split into 9 parts by 4 segments with one part be rectangle. It is known that areas off 8 parts (except one part that is' triangle') is expressed by rational numbers. Could it happen that the area of the last part is irrational?

SOLUTION. Answer: NO Assume that the aria in mention is irrational. Let us construct four chords, symmetrical to the initial set of chords with respect to the center.

9. For parabola $y = x^2$ a sequence of the circles is defined as follows: $C_1$ with radius $1/2$ is tangent to the parabola at vertex; each $C_n$ $(n > 1)$ is tangent to parabola and $C_{n-1}$. Evaluate the radius of $C_{2004}$.

   SOLUTION. Let $r_n$ be the radius of $C_n$ and $h_n = r_1 + r_2 + \cdots + r_n$. Then the circle $C_n$ is described by equation $x^2 + (y - (2h_{n-1} + r_n))^2 = r_n^2$.

   The fact that $C_n$ is tangent to the parabola implies that equation $y + (y - (2h_{n-1} + r_n))^2 = r_n^2$ has one double root and thus its discriminant is equal to 0:

$$\Delta = (4h_{n-1} + 2r_n - 1)^2 - 16h_{n-1} \times (h_{n-1} + r_n) = (2r_n - 1)^2 - 8h_{n-1} = 0.$$

   Then $r_n = \frac{\sqrt{8h_{n-1}}+1}{2}$.

   One can prove by induction that $r_n = n - \frac{1}{2}$ and $h_n = \frac{1}{2}n^2$. Thus $r_{2004} = 2003\frac{1}{2}$.

10. There is the full set of domino in a box. Two players in turns play the following game: each player chooses a piece of domino from the box and adds it to the chain according to domino rules. The player who cannot move loses. Who of the players has a winning strategy?

    SOLUTION. Let us describe the winning strategy for FP. On his first move he places $0 - 0$; SP responds with $0 - a$ (with no loss of the generality $0 - 1$) then FP places $1 - 1$.

    Now the strategy of FP is to keep chain (after each of his moves) with ends 0 and 1. If SP places $0 - b$ then $FP$ responds by $b - 1$ and if SP places $1 - b$ then FP responds by $b - 0$. Since there are five $0 - b$ and five $1 - b$ pieces with $b \neq 0, 1$ and SP is the first to use them, then FP has the last piece.

11. On coordinate axis at one of the marked points (with integer coordinates from 1 to 2004) sits Grasshopper. After 2004 jumps it visited all mark points and returned to the point it started. It is known that any two subsequent jumps have opposite directions and the total sum of the lengths of all the jumps (except the last one) is equal to 4005. Find the length of the last jump.

    SOLUTION. We start from

    **Proposition 1.** *Let the numbers from 1 to $2n$ be placed on the circle in such a way that each number is either larger (call it red) or smaller (call it blue) than both of its neighbors . Then the difference between the total sum of all red numbers and the total sum all of blue numbers is no less than $3n - 2$.*

    *Proof.* By induction. □

    Applying to our problem we conclude that the total sum of all Grasshopper jumps is at least $6n - 4 = 6008$ and therefore the last jump is at least 2003. Since it cannot exceed 2003, the last jump is exactly 2003.

12. Let $n$ and $m$ be positive integers, $n > 1$. It is known that the number $m^2n^2 - 4m + 4n$ is a perfect square. Find the possible relations between $m$ and $n$.

Solution. We prove that $m = n$. Let $K_{mn} = m^2n^2 - 4m + 4n$. If $2 \leq m < n$ then

$$(mn)^2 < K_{mn} < (mn + 1)^2$$

and $K_{mn}$ is not a perfect square.

If $m = 1, n > 2$ then

$$(n + 1)^2 < K_{1n} < (n + 2)^2;$$

is not a perfect square; $K_{12} = 8$ is not a perfect square either.

If $m > n$ then

$$\left(mn^2 - 2\right)^2 < n^2K_{mn} < m^2n^4;$$

then $n^2K_{mn} = \left(mn^2 - 1\right)^2$ or $4n^3 = 2mn^2 + 1$ which is impossible.

So, $m = n$. On the other hand, for $m = n$ $K_{mn} = m^2n^2$ is a perfect square.

13. For any two distinct elements a and b from set M (M is a set of real numbers, consisting of 2004 distinct elements) the number $a^2 + b\sqrt{2}$ is rational. Is it true that for any element of any such M the number $a\sqrt{2}$ is rational?

Solution. Yes, it is. Let us consider any three distinct elements a,b,c from M. Since numbers $a^2 + b\sqrt{2}$, $b^2 + a\sqrt{2}$, $c^2 + a\sqrt{2}$, $c^2 + b\sqrt{2}$ are rational, then numbers $a^2 + b\sqrt{2}$-( $b^2 + a\sqrt{2}$)=$(a - b)(a + b - \sqrt{2})$ and $c^2 + a\sqrt{2} - c^2 + b\sqrt{2}$=$a\sqrt{2} - b\sqrt{2}$ are rational. Then $a\sqrt{2} + b\sqrt{2}$ is rational as well. Therefore, the number $a\sqrt{2}$=$1/2(a\sqrt{2} + b\sqrt{2} + a\sqrt{2} - b\sqrt{2})$ is rational.