MathBattle 3: Season 2003/2004: Solutions

April 18 , 2004

1. Let A and B be two given points on the circumference. Find locus of midpoints of chords (of given circumference) which intersect the segment AB.

   SOLUTION. Let O be the center of the circle. In other words, we need to find the locus of points M, such that straight line ℓ perpendicular to OM intersects AB.

   Notice, that ℓ passes through points A (B), when $\angle AMO$ ($\angle BMO$) is a right angle. It is easy to see that ℓ intersects AB iff exactly one of angles $\angle AMO$ and $\angle BMO$ is obtuse. Therefore locus in mention consists of the points, which belong to one of the circles with diameters OA or OB (but not both).

2. One need to sew up a parcel (in a shape of a cube with the edge 1) in canvas (in one layer). Could it happen that the total length of the seam is less than 6.5?

   ANSWER. Yes. Let coordinates of vertices of the cube be

   $$A(0,0,0); B(1,0,0); C(1,1,0); D(0,1,0);$$
   $$A'(0,0,1); B'(1,0,1); C'(1,1,1); D'(0,1,0).$$

   Consider points $M(a,1/2,0)$; $N(1-a,1/2,0)$; $M'(a,1/2,1)$; $N'(1-a,1/2,1)$ where $a=1/2\sqrt{3}$. Then the total length of the seam is

   $$AM+DM+MN+BN+NC+DD'+A'M'+D'M'+M'N'+B'N'+N'C' = 3+2\sqrt{3}=6.465\ldots. < 6.5.$$

3. During a joined conference 32 representatives of two parties (Truthtellers who always tell the truth and Liars who always lie) were occupying 4 rows, eight sits each. After the conference every representative claimed that among his neighbors there were representatives of both parties. Find the minimal number of Liars for this situation be possible. (Two people are neighbors if one is sitting to the left, right, behind or ahead of the other).

   ANSWER: 8. Let us consider 4×8 table, divided into 8 regions:

   $$\{(1,1),(1,2),(2,1)\},$$
   $$\{(3,1),(3,2),(4,1),(4,2)\},$$
   $$\{(1,3),(2,2),(2,3),(2,4),(3,3)\},$$
   $$\{(3,4),(4,3),(4,4),(4,5)\},$$
   $$\{(1,4),(1,5),(1,6),(2,5)\},$$
   $$\{(2,6),(3,5),(3,6),(3,7),(4,6)\},$$
   $$\{(1,7),(1,8),(2,7),(2,8)\},$$
   $$\{(3,8),(4,7),(4,8)\},$$

   with Liars placed on (1,2), (1,6), (2,4), (2,8), (3,1), (3,5), (4,3), (4,7). The numbers of Liars can not be less than 8; for in this case one of the regions is occupied only by Truthtellers , which is impossible.

4. Let $P(x)$ be a polynomial with real coefficients. Given that $x^2 + x + 1$ divides $P(x^3)$, find a root of $P(x)$.

   ANSWER: $x = 1$.

   SOLUTION. Note that $P(y) - P(1)$ is divisible by $(y-1)$ and therefore $P(x^3) - P(1)$ is divisible by $(x^3 - 1)$ and thus it is divisible by (x^2+x+1). Since $P(x^3)$ is also divisible by (x^2+x+1), we conclude that $P(1)$ is divisible by $(x^2 + x + 1)$ as well; then $P(1) = 0$.

5. Is it possible to inscribe a square into a regular pentagon, so that all vertices of the square belong to the boundary of the pentagon?

   SOLUTION. Let us draw a line MN parallel to the side AB of regular polygon $ABCDE$. Let K, R and L be points of intersection of perpendiculars at points M, A, N to side AB with sides ED, ED and DC correspondingly. Then $MKLN$ is an inscribed rectangle. Let $x = MN$ and $y = MK$. Let us slide $MN \parallel AB$ until it reaches position EC. Then x increases from AB to EC, y continuously decreases from AR to 0. One can calculate $AR = 2\cos 54° > 1$ and $EC = 2\cos 36° > 1$ (assuming that $AB = 1$). Then there exists x such that $y(x) = x$ and a rectangle becomes a square.

6. Let S be any set of 2^{n+1} positive integers. Is it always possible to choose from S a subset consisting of exactly 2^n elements, such that their sum is divisible by 2^n?

   ANSWER: yes.

   SOLUTION. Proof by induction. Base $n = 1$ is obvious. Let us split S into halves, each containing 2^n elements. In each half let us choose a subset of 2^{n-1} positive integers with their sum divisible by 2^{n-1}. From what is left (2^n numbers) again choose a set of 2^{n-1} positive integers with their sum divisible by 2^{n-1}. Let sums of these three sets be $2^{n-1}a$, $2^{n-1}b$, $2^{n-1}c$. Among a, b, c there are two numbers of the same parity. Join these two sets together. Then a set constructed contains 2^n elements with their sum divisible by 2^n.

7. Two players in turns are moving a Knight on 2004×2004 chessboard. On his first turn FP (First Player) places the Knight on the square of his choice and then moves it.

   FP is allowed to make only *horizontal moves* (the Knight moves to the neighboring horizontal: up-right-right, or down-right-right, or up-left-left, or down-left-left) while SP is allowed to make only *vertical moves* (the Knight moves to the neighboring vertical: left-up-up, or right-up-up, or left-down-down, or right-dow). It is not allowed to place the Knight on the square it has visited before. Player who has no move loses. Who of the players has a winning strategy?

   SOLUTION. FP has a winning strategy.

   First note that the answer would be the same if players interchange types of their moves. Assume that FP has no winning strategy; then SP has it for every possible first move of FP, since there is no draw.

   Let FP start placing the Knight on a square x (assume x is black) and moving it to a square y. Let us consider a copy of the board with Knight placed on y. When SP moves the Knight to some square (say z) on the first board, FP mirrors his move on the second board. Then FP (who is playing as the second player on the second board) using the winning strategy (for the second player) to make his move there and then mirrors it on the first board and so on.

   Note that FP playing on the second board cannot lose since he applies the winning strategy. Since all moves (except the first one on the first board) are the same, then FP cannot lose on the first board unless he is forced to move the Knight on the initial square. However this is impossible: since after each move the color of the square changes and FP always moves from black to white. So FP cannot lose.

8. Let us consider all possible functions $f(x) = ax^2 + bx + c$ $(a < b)$ such that $f(x) \geq 0$ for all x. Find the minimal value of $(a + b + c)/(b - a)$.

ANSWER: 3.

SOLUTION. Condition $f(x) \geq 0$ for all x means that $a > 0$ and $b^2 - 4ac \leq 0$ or $c \geq b^2/4a$; given that $b > a$ it implies that $c > a/4$. Then

$$A = \frac{a+b+c}{b-a} = 1 + \frac{2a+c}{b-a} \geq 1 + \frac{2a+c}{2\sqrt{ac}-a} = g(t) \stackrel{\text{def}}{=} 1 + \frac{2t+\frac{1}{t}}{2-t}$$

where $t = \sqrt{\frac{a}{c}} \in (0, 2)$. One can show that $g(t)$ reaches its minimum equal 3 on $(0, 2)$ at $t = \frac{1}{2}$. Thus $A \geq 3$. On the other hand $A = 3$ for $c = b = 4a$.

9. Fibonacci numbers are defined by $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$ $(n = 1, 2, \ldots)$. Prove that the sum of their inverse numbers is less than 4.

SOLUTION. Let

$$S_N = \sum_{n=1}^{N} \frac{1}{a_n} = 1+1+\sum_{n=3}^{N} \frac{1}{a_n} = 2+\sum_{n=1}^{N-2} \frac{1}{a_n + a_{n+1}} < 2+\sum_{n=1}^{N-2} \frac{1}{2a_n} < 2+\frac{1}{2}S_N \implies S_N < 4 \quad (*)$$

for all N and therefore $\sum_{n=1}^{\infty} a_n^{-1} < \infty$ ($\{S_N\}$ is bounded and increasing) and then all series in (*) are converging and one can repeat all arguments there with $N = \infty$.