MathBattle 1: Season 2003/2004: Problems & Solutions

December 14 , 2003

1. Let $a_1 = 1$, $a_2 = 6$, $a_{n+2} = a_{n+1} + b_{n+1}$, where $b_{n+1} = \begin{cases} \sin a_{n+1} & \text{if } a_{n+1} > a_n \\ \cos a_{n+1} & \text{if } a_{n+1} \le a_n \end{cases}$

   Prove, that $a_n < 10$ for any positive integer n .

   Solution: We prove that if $a_{n+1} < 3\pi$ then $a_{n+2} < 3\pi$. It follows from the fact that both functions $f(x) = x + \sin x$ and $g(x) = x + \cos x$ are monotone increasing and $f(3\pi) = 3\pi$, $g(3\pi) = 3\pi - 1 < 3\pi$.

   Now $a_n < 3\pi < 10$ by induction.

2. On MathBattle Tournament $2N$ teams are registered for participation. The Organizing Committee decided:

   (a) Every two teams play exactly once between themselves during the season.

   (b) All teams play in each round.

   However, having problems with the other issues, the Organizers schedule each new round randomly; just that on this particular round both conditions (a),(b) hold.

   Could it happen, that one day (on a particular round) the Committee would not be able to make a schedule satisfying all the requirements? Consider the following cases:

   (a) $N = 5$;

   (b) $N = 6$;

   (c) N is arbitrary.

   Solution. Lets call number N Bad, if it could happen that on some round conditions (a) and (b) are impossible to fulfill and *Good* otherwise.

   Obviously, $N = 1, 2$ are Good numbers. Let us prove, that any $N > 2$ is Bad.

   Lemma 1. Let us consider two disjoint groups, k teams in each. Then it is possible to scheduled first k rounds, so that each team of the first group play with each team of the second group.

   Proof. Let $1, 2, \ldots, k$ and $(k+1), (k+2), \ldots, (2k)$ be numbers of the teams of the first and the second group respectively. Let pair team s and team t in round m iff $t - s \equiv m \mod k$ $(m = 0, 1, \ldots, k-1;\ 1 \le s \le k;\ 1 + k \le s \le 2k)$.

   Lemma 2. Any odd number $N > 1$ is Bad.

   Proof. According to Lemma 1 after N rounds it could happen that each team $1, \ldots, N$ played already with each team $(N+1), \ldots, (2N)$ and it is necessary to divide N teams in pairs. However, N is odd.

   Lemma 3. Number 4 is Bad.

   Example:

   Round 1. 1–4; 2–5;3–6; 7–8.

   Round 2. 1–5; 2–6;3–7; 4–8.

Round 3. 1–6; 2–7;3–8; 4–5.

Round 4. 1–7; 2–8;3–4; 5–6.

Round 5. 1–8; 2–4;3–5; 6–7.

Then, on round 6 teams 1,2, and 3 could play only between themselves; however, it is impossible to arrange.

Lemma 4. If N is Bad, then $2N$ is also Bad.

Proof follows from Lemma 1.

From lemmas 2–4 it follows that any even $N > 2$ is Bad.

3. A circle and a point outside of it are given. Using only a straight edge, construct a tangent to the given circle.

Solution. Let us draw three secant lines ℓ_1, ℓ_2, ℓ_3 from the given point (P). Let A_j, B_j be points of intersection of the circle $(\mathcal{C})$ and secant ℓ_j. Let us find O_1, O_2 points of intersection of A_1B_2 with A_2B_1 and A_2B_3 with A_3B_2 respectively. Then points M_1, M_2 of intersection of line O_1O_2 and $\mathcal{C}$ are points of tangency.

Justification follows from

Theorem. Two tangents and two secant lines are drawn to a circle. Then the point of intersection of the diagonals of a quadrilateral created by points of intersection of secant lines with the circle belong to the segment connecting points of tangency.

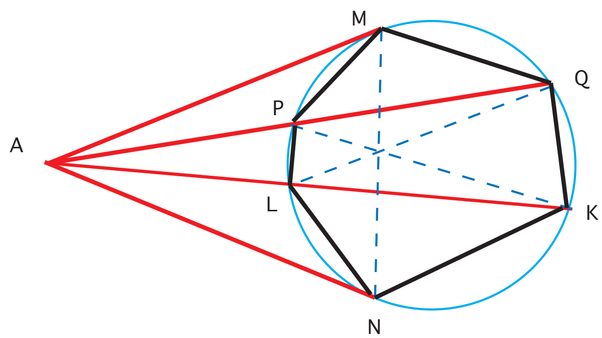
Proof. We use Ceva's theorem: Consider $\triangle ABC$ and three lines AA_1, BB_1, CC_1 where A_1, B_1, C_1 lie on sides BC, AC and AB respectively. Then AA_1, BB_1 and CC_1 intersect at the same point if and only if

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1, \tag{1}$$

or, (the other form of the condition):

$$\frac{\sin \angle ACC_1}{\sin \angle C_1CB} \cdot \frac{\sin \angle BAA_1}{\sin \angle A_1AC} \cdot \frac{\sin \angle CBB_1}{\sin \angle B_1BA} = 1. \tag{2}$$

Proposition. Let $PMQKNL$ be an inscribed hexagon.



Then the main diagonals PK, NM, QL intersect at the same point if and only if

$$\frac{PM}{MQ} \cdot \frac{QK}{KN} \cdot \frac{NL}{LP} = 1. \tag{3}$$

Proof follows from Theorem and the fact that each chord equals the diameter multiplied by sinus of the corresponding inscribed angle.

Proof of Theorem (end). From similarity of triangles:

$\triangle APM \sim \triangle AMQ \implies$

$$\frac{PM}{MQ} = \frac{AM}{AQ}, \tag{4}$$

$\triangle APL \sim \triangle AKQ \implies$

$$\frac{QK}{PL} = \frac{AQ}{AL}, \tag{5}$$

$\triangle ALN \sim \triangle ANK \implies$

$$\frac{LN}{NK} = \frac{AL}{AN}. \tag{6}$$

Multiplying (4),(5),(6) and noticing that $AM = AL$ we get (3).

4. There are k nails in the wall and a picture with a string attached to it (in a usual way: each end of the string is attached to one corner of the picture). Is it possible to hang this picture in such a way that it would fall down if either of the nails is removed?

   (a) $k = 2$;

   (b) $k = 3$.

   Solution. Yes. Let A, B, C be nails. Let us draw up a vertical ray α from A. Let us move along the string from left to the right end. If during this movement ray α is crossed from left to the right, denote this event by A^+, and by A^- otherwise. Similarly we introduce $B^\pm, C^\pm$. Note, that K^+K^- or K^-K^+ $(K = A, B, C)$ means crossing forward and backwards, which is equivalent to no crossing at all.

   Method to hang:

$$A^+B^+A^-B^-C^+B^+A^+B^-A^-C^-.$$

   One can see that

   (a) the picture really holds and

   (b) by removing A we get $B^+B^-C^+B^+B^-C^-$, which is equivalent to $C^+B^+B^-C^-$, equivalent to C^+C^-, equivalent to no crossing at all.

   The same holds if we remove either B, or C.

   For $k = 2$ the method to hang: $A^+B^+A^-B^-$.

5. On the paper sheet 64 points are given. Two players in turns play the following game. First Player connects any two points that are not yet connected. Second Player responses by placing an arrow on this edge. Second Player wins if after 1959 turns it is possible to reach any point from any other point, moving along arrows; otherwise, First Player wins. Who has the winning strategy?

   Solution. First Player wins.

Let's notice, that the number $1959 = 63 \times 62/2 + 6$ is by 6 greater than the number of the edges of the complete graph (each pair of vertices is connected) with 63 vertices.

Let's describe First Player's strategy. First, he splits the set of 64 vertices into 32 pairs, and connects every pair (on his 32 turns).

After Second Player's responses he has 32 vertices with 1 entering and 0 exiting edges.

Now, First Player splits the set of these 32 vertices into 16 pairs, and connects every pair. After Second Player's responses he has 16 vertices with 2 entering and 0 exiting edges.

Continuing in the similar way, First Player gets a vertex (V), that with 6 entering and 0 exiting edges. After that, First Player connects all vertices but V to make the graph complete. As a result, he has an oriented graph with 1959 edges and vertex V, from which no edge exits and thus one cannot reach any other vertex from it.

6. Mr. Bean decided to cover an ugly spot on a wall in a shape of equilateral triangle by the artifact exactly of the same shape and size ($\triangle ABC$) . In order not to spoil the artifact he is using nails merely as supports. He already nailed nails K and L (nails are points on sides of the triangle ($AK : KB = 2 : 1$, $BL : LC = 3 : 2$). Now, he is deciding where he should nail the last nail M in order to fix the artifact rigidly (so it could not be moved). Help Mr. Bean to make a decision.

Solution. We start from

Theorem. Triangle is fixed if and only if the perpendiculars to AB, BC, CA at points K, L, M respectively intersect at the same point.

Proof. Note first that if P is a point inside of half-plane $\mathcal{H}$ and M is a point on its boundary ℓ and if $PM \not\perp \ell$ then $\mathcal{H}$ could be slightly rotated around P without covering M (in the direction that PM goes away from PP' ($PP' \perp \ell$)). It is possible, since the rotation in mention is equivalent to rotation of M around P in the opposite direction when $\mathcal{H}$ remains fixed.

On the other hand, if $PM \perp \ell$ then such rotation is impossible.

Now, if the perpendiculars to the side of triangle do not intersect at the same point, they form a polygon $\mathcal{F}$. Then we can rotate $\triangle ABC$ around any inner point P of $\mathcal{F}$ (all rotations described above are in the same direction).

Let P be an intersection point of all three perpendiculars. Assume that all sides of $\triangle ABC$ equal 15 (units). Then $AK = 10$, $KB = 5$, $BL = 9$, $LC = 6$, $AM = x$ and $CM = 15 - x$. We have

$$AP^2 - AK^2 = PK^2 = BP^2 - BK^2.$$

Then

$$AP^2 - BP^2 = AK^2 - BK^2 = 75.$$

In a similar way we get

$$BP^2 - CP^2 = BL^2 - CL^2 = 45$$

and

$$CP^2 - AP^2 = CM^2 - AM^2 = 225 - 30x.$$

Adding up these three equations we get

$$75 + 45 + 225 - 30x = 0 \implies x = \frac{23}{2}.$$

Therefore $AM : MC = 23 : 7$.

7. Find the number of solutions for the following equation:

$$\log_{\frac{1}{16}} x = \left(\frac{1}{16}\right)^x.$$

Answer: 3.

Solution. Consider $y = f(x) = \log_{\frac{1}{16}} x$ and $y = g(x) = \left(\frac{1}{16}\right)^x$. These are inverse functions,monotone with domains $(-\infty, \infty)$ and $(0, \infty)$ respectively and thus their graphs intersect on the bisector $\ell = \{x = y > 0\}$ (point of intersection corresponds to $x = \left(\frac{1}{16}\right)^x$)
One can find two other solutions $(x, y) = (\frac{1}{4}, \frac{1}{2})$ and $(x, y) = (\frac{1}{2}, \frac{1}{4})$, symmetric with the respect to ℓ.

Let us prove that there are no other solutions. Rewrite equation as

$$F(x) = \log_{\frac{1}{16}} \left(\log_{\frac{1}{16}} x\right) - x = 0 \qquad (*)$$

Assumption that (*) has more than 3 roots implies that $F'(x) = 0$ has more than 2 roots:

$$\frac{1}{(\ln 16)^2} \cdot \frac{1}{x \ln x} = 1$$

or

$$G(x) = x \ln x - (\ln 16)^{-1} = 0. \qquad (**)$$

However $G''(x) > 0$; then the graph of $G(x)$ is convex and therefore (**) has no more than 2 roots.

8. There are 9 coins, all alike, but one is fake (lighter, than the real one). Also there are two balances: one is precise and the other is rough (can not show the difference between real and fake coins). Find the minimal number of weightings one needs to determine a fake coin if we do not know which of the balances is precise.

Answer: 3.

Solution. Number of weighings cannot be less then 3; 2 weightings are needed to determine a fake coin in the problem in the case of one precise balance.

1. Using the first balance compare weights of two groups of 4 coins each.
   (a) One side is lighter. We conclude that the balance is precise and the fake coins is in a lighter group. One can find it using two more weightings.
   (b) Balance is in equilibrium. Then either the first balance is precise (then the 9-th coin is fake), or the second balance is precise. Then we continue with
2. Using the second balance compare weights of two groups of three coins each; namely three coins from the first group and three coins from the second group.
   (a) One side is lighter. We conclude that the second balance is precise and a fake coin is in a lighter group of three. One more weighting is enough to determine a fake coin.
   (b) Balance is in equilibrium. Then either the first balance is precise (then the 9-th coin is fake), or the second balance is precise (then a fake coin could be either one of two "leftovers" or the 9-th coin). In either case we proceed with
3. Using the second balance compare weights of two "leftovers".
   (a) One side is lighter, so we determine a fake coin.
   (b) Balance is in equilibrium. Therefore the both groups of 4 coins are real (confirmed by first and the second balances). Then the 9th coin is a fake.