MathBattle 2: Season Opening.

December 14 , 2003 Problems.

1. Let $a_1 = 1$, $a_2 = 6$, $a_{n+2} = a_{n+1} + b_{n+1}$, where $b_{n+1} = \begin{cases} \sin a_{n+1} & \text{if } a_{n+1} > a_n \\ \cos a_{n+1} & \text{if } a_{n+1} \leq a_n \end{cases}$

   Prove, that $a_n < 10$ for any positive integer n .

2. On MathBattle Tournament $2N$ teams are registered for participation. The Organizing Committee decided:

   (a) Every two teams play exactly once between themselves during the season.

   (b) All teams play in each round.

   However, having problems with the other issues, the Organizers schedule each new round randomly; just that on this particular round both conditions (a),(b) hold.

   Could it happen, that one day (on a particular round) the Committee would not be able to make a schedule satisfying all the requirements? Consider the following cases:

   (a) $N = 5$;

   (b) $N = 6$;

   (c) N is arbitrary.

3. A circle and a point outside of it are given. Using only a straight edge, construct a tangent to the given circle.

4. There are k nails in the wall and a picture with a string attached to it (in a usual way: each end of the string is attached to one corner of the picture). Is it possible to hang this picture in such a way that it would fall down if either of the nails is removed?

   (a) $k = 2$;

   (b) $k = 3$.

5. On the paper sheet 64 points are given. Two players in turns play the following game. First Player connects any two points that are not yet connected. Second Player responses by placing an arrow on this edge. Second Player wins if after 1959 turns it is possible to reach any point from any other point, moving along arrows; otherwise, First Player wins. Who has the winning strategy?

6. Mr. Bean decided to cover an ugly spot on the wall in a shape of equilateral triangle by the artifact exactly of the same shape and size ($\triangle ABC$) . In order not to spoil the artifact he is using nails merely as supports. He already nailed nails K and L (nails are points on sides of the triangle ($AK : KB = 2 : 1$, $BL : LC = 3 : 2$). Now, he is deciding where he should nail the last nail M in order to fix the artifact rigidly (so it could not be moved). Help Mr. Bean to make a decision.

7. Find the number of solutions for the following equation:

$$\log_{\frac{1}{16}} x = \left(\frac{1}{16}\right)^x.$$

8. There are 9 coins, all alike, but one is fake (lighter, than a real one). Also there are two balances: one is precise and the other is rough (can not show the difference between real and fake coins). Find the minimal number of weightings one needs to determine the fake coin if we do not know which of the balances is precise.