MathBattle 1: Season 2003/2004: Solutions

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1. Hydra consists of heads and necks (each neck connects exactly two heads). By one blow of his sword Heracles can cut all the necks connected to any one head. At the same moment, however, new necks grow instantaneously from this head to all the heads, unconnected previously with the head in mention. Heracles defeats hydra, if he separates it into two unconnected parts. Find the minimal number of blows that would allow Heracles to defeat any 100-neck hydra.

   Answer: 10.

   Solution.

   (i) Consider a graph, in which vertices represent heads and necks represent edges. Then the blow is an inversion of vertex A: we remove all existing edges emanating from A and draw all the other edges from it.

   (a) Assume that there is a vertex A with degree (number of edges emanating from it) not exceeding 10. Then A is separated by inversion of all its neighbors.

   (b) Assume that there is a vertex, connected to all the heads but n ($n < 10$). Then inverting this vertex first, we get case (a).

   (c) Case, when each vertex has degree at least 11 and at the same time not connected with at least 10 vertices is impossible. Really, the number of vertices is at least 22, meaning that the number of edges is not less than $22 \times 11 > 100$.

   (ii) Example of Hydra, which can not be defeated with 9 blows. There are two groups of 10 vertices each; each pair of vertices of different groups is connected (by 100 edges altogether). Really, if the number of blows does not exceed 9, then two groups are still connected (by at least one edge).

   At the same time inside of each group, all "odd" heads are connected to "even" ones: head is "odd" ("even") if it is cut off the odd (even) number of times.

2. Solve

$$\begin{cases} x = \sqrt{yz}/(y+z) \\ y = \sqrt{zx}/(z+x) \\ z = \sqrt{xy}/(x+y) \end{cases}$$

   Answer: $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$; $(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$.

   Solution.

   Let (x, y, z) solve the system. Notice, that x, y, z have the same sign. Assume that $x, y, z > 0$. Since $(y+x)/2 \geq \sqrt{yx}$ then $x \leq \frac{1}{2}$ ($y \leq \frac{1}{2}$, $z \leq \frac{1}{2}$). On the other hand, multiplying all three equations, we get

$$(y+z)(z+x)(x+y) = 1$$

   and therefore $x = y = z = \frac{1}{2}$. If x, y, z are negative we can replace them by $-x, -y, -z$.

3. Quadrilateral with perpendicular diagonals is inscribed into a square (so that each side of the square contains exactly one vertex of the quadrilateral). Each of 8 triangles created is colored in Red or Blue; moreover, triangles that share a side, are painted in different colors. Could it happen that sums of radiuses of circles inscribed into Red and Blue triangles are different?

   Answer: No.

Solution.

Proposition 1. $d = a + b - c$, where a, b, c are lengths of sides right angle triangle and d is a diameter of an incircle.

Proposition 2. Let square be cut by two perpendicular segments into four quadrilaterals. Then sums of the perimeters of quadrilaterals with no adjacent sides are the same.

Proof. After parallel translation of each segment such that their point of intersection translates to the center of square, the proof becomes easy.

Now, from proposition 2 it follows that the sums of legs of all Red and all Blue triangles are the same. Notice that it also true for hypothenuses of Red and Blue triangles. Then, using proposition 1 we conclude that two sums in mention are the same.

4. In interval (n^2, n^2+n), where n is a given positive integer, two distinct positive integers A and B are chosen. For the number $A \cdot B$, find all positive divisors from the interval in mention.

   Solution. Let $d, n^2 < d < n^2 + n$ be divisor of AB. Since d divides $A \cdot B$ then

$$d|(A-d)(B-d) \qquad (1)$$

   Since A, B, d belong to the same interval of the length n, we have : $|A - d| < n, |B - d| < n$. Therefore,

$$|(A-d)(B-d)| < n^2 < d. \qquad (2)$$

   From (1) and (2) we have $(A - d)(B - d) = 0$. Then the possible values for d are A and B.

5. $(10N)$ worms are placed into N jars. For any two jars with even total number of worms it is allowed to put worms from one jar to the other equalizing numbers of worms in both jars. Applying this operation as many times as one wishes is it always possible to get the same number of worms in all jars?

   (a) $N = 8$.
   (b) $N = 10$.

   Solution. (a) Yes.

   First, let us consider the case: k jars with $2r$ worms, and $(8 - k)$ jars with $(2s + 1)$ worms in each.

   Then we have $80 = 2kr + (8-k)(2s+1)$ or $72-16s = k(2r-2s-1)$, meaning that 16 divides k. Then $k = 0$ and $10 = 2s + 1$ which is impossible.

   Therefore, there two jars with the numbers of worms distinct, but the same parity (unless we already have the same number of worms in all jars).

   We equalize the number of worms in these jars. Then we either get the same number of worms in all jars or we'll proceed further. The process stops, since in result of each operation $a^2 + b^2$ is changed to $(a + b)^2/2$, where a and b are numbers of worms in chosen jars (and $a^2 + b^2 > (a + b)^2/2$ if $a \neq b$).

   (b) No.

   Consider 8 jars with 11 worms and 2 jars with 6 worms in each.

6. An equilateral triangle ABC is given. Find a locus of points M, such that both triangles ABM and ACM are isosceles.

   ANSWER. Circumference $\mathcal{C}$ (with the center A and radius AB), without four points (C, B and their reflections with respect to A) plus six more points (M_1, the center of $\triangle ABC$; M_2 and M_3, points of intersection a bisector of $\angle ABC$ and circumference $\mathcal{C}_1$ (with the center B and the radius AB); M_4 and M_5, points of intersection a bisector of $\angle ACB$ and circumference $\mathcal{C}_2$ (with center C and radius AC); M_6, point of intersection of circumferences $\mathcal{C}_1$ and $\mathcal{C}_2$, distinct from A).

7. A segment AB and a line ℓ , parallel to it are given. Using only a straight edge, divide the segment into halves.

   SOLUTION. Let M be a point outside of the stripe created by ℓ and extension of AB. Draw lines MA and MB from some point M. Let C and D be their intersections with ℓ. Draw CB and AD. Let N be their intersection. Then MN divides AB into halves. Justification follows from similarity of triangles.

8. There is a 1×2003 board. Two players in turns place a black or white checker on an empty square. Player, who places a checker next to a checker of different color, loses. If nobody loses, the game is considered to be a draw. Who has a winning strategy? (There are at least 2003 checkers of each color in the stock).

   ANSWER. Nobody.

   SOLUTION. SP counters each move of FP (unless it is the move on the central square) by placing a checker of the opposite color on symmetric square (with respect to the center).

   If FP places a checker on the central square, then SP places a checker of the same color next to it (this move is possible without losing). Then appears a "central core" consisting of row of checkers of the same color, which lacks one checker to be symmetric. If FP symmetrizes the central core, SP makes it asymmetric again (it is always possible without losing); otherwise SP counters FP's move by placing a checker of opposite color on symmetric square.

   Then either FP cannot place a checker without losing unless all board is filled with the checkers of the same color. So, the SP has a draw for sure.

   However, SP cannot win against the following strategy of FP: FP places a checker in the center and then counters each move of SP by placing a checker of the same color on symmetric square.