MathBattle Superevent: Problems

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1. Last summer Joe was in the summer camp together with 44 other students. When the camp was over, 950 pairs of students exchanged their addresses. Later Joe wished to write a letter to Jane whose address he did not know. Will Joe always be able to communicate with Jane? In other words: does there always exist a chain of students from Joe to Jane, in which every two consecutive students exchanged their addresses?

Solution. Let us consider graph with 45 vertices (students) and 950 edges (address exchanges). We need to prove that this graph is connected. To the contrary, consider unconnected graph containing the largest number of edges. This is a graph, consisting of two components, in which all pairs of vertices are connected. Let larger component contain k vertices, $k = 23, \ldots, 44$. Then the total number of edges is $\frac{1}{2}\big(k(k-1) + (45-k)(44-k)\big) = k^2 - 45k + 45 \cdot 22$ which reaches its maximum $946 < 950$ as $k = 44$. Contradiction.

2. In the school Chess Club students can play chess either between themselves or with a computer. Last meeting the Club was attended by n students. Each of them played no more than n games, and any two of them who did not play with each other, played altogether no more than n games. Prove that all the students altogether played no more than $n(n+1)/2$ games.

Solution. Proof by induction. Base $n = 1$ is obvious.

Let us assume that for $n-1$ students our statement is true and consider n students. Let A be a player who played most games. If A played no more then $(n+1)/2$ games, so did everybody and the total number of games does not exceed $n(n+1)/2$.

Assume that A played more than $(n+1)/2$ games. Let call the game *essential* if A did not play it. It is enough to prove that there are no more than $n(n-1)/2$ essential games. The latter follows from induction if we prove that the set of participants without A satisfies our conditions. Really: if B played with A, then B played no more than $n-1$ essential games and if B did not play with A, he played less than $n - (n+1)/2 = (n-1)/2$ games. Let B, C did not play between themselves. If none of them played with A, then together they played less than $(n-1) = (n-1)/2 \ + \ (n-1)/2$ games. If at least one of them played with A, they played together no more than $n-1$ essential games.

3. There is a set of weights of total weight $2s$. Let's call a natural number k *admissible* if one can chose k weights from the set of total weight s. What is the maximal quantity of admissible numbers when the set consists of 100 weights?

Solution. Let us notice that 100 is not an admissible number, that if k is admissible then $100 - k$ is admissible as well and that if $k = 1$ is admissible then the only other admissible number is 99. Then there are no more than 97 admissible numbers.

Example of the set with 97 admissible numbers: Let us set $a_1 = a_2 = 1, a_n = a_{n-1} + a_{n-2}$ for $n = 3, \ldots, 99$ and $s = a_1 + a_2 + \ldots + a_{98}$, $a_{100} = s - a_{99}$. Then

$$s = a_{100} + a_{99} = a_{100} + a_{98} + a_{97} = \ \ldots \ = a_{100} + a_{98} + a_{96} + \ldots + a_4 + a_3 + a_2.$$

Therefore $2, 3, \ldots, 51$ are admissible numbers and so $a_{98}, a_{97}, \ldots, a_{52}$ are admissible as well.

4. The King is placed on some square of the 8×8 chessboard. Two players in turns move it according to the standard rules. It is forbidden to reverse a previous move. The player wins if he moves the King on a square which the King visited before. Who has a winning strategy?

Solution. The first player wins. Assume without any loss of generality that the King can reach the bottom horizontal in odd number of moves. Then the first player moves the King strictly down. Then the second player is forced to move down vertically or diagonally. Since the first player is the first to reach the bottom, he wins.

5. The equation $x^3 + ?x^2 + ?x + ? = 0$ is written on the board. The first player names a number and the second replaces one of "?"s by it. The same procedure is repeated twice more. The goal of the first player is to get an equation with three disjoint integer roots. Can the second player prevent him?

Solution. No, second player cannot. In his first move the player calls 0. We have three cases to consider:

(a) $x^3 + ?x^2 + ?x = 0$. First player calls 2 and then -3.

(b) $x^3 + ?x + ? = 0$. First player calls $-(3 \cdot 4 \cdot 5)^2$. Then

(b1) $x^3 - (3 \cdot 4 \cdot 5)^2 x + ? = 0$. First player calls 0.

(b2) $x^3 + ?x - (3 \cdot 4 \cdot 5)^2 = 0$. First player calls $3^2 \cdot 4^2 - 3^2 \cdot 5^2 - 4^2 \cdot 5^2$.

(c) $x^3 + ?x^2 + ? = 0$. First player calls $6^2 \cdot 7^3$.

(c1) $x^3 + ?x^2 + 6^2 \cdot 7^3 = 0$. First player calls -7^2.

(c2) $x^3 + 6^2 \cdot 7^3 x + ? = 0$. First player calls $-6^8 \cdot 7^6$.

6. There are n barrels containing 1 L, 2 L, $\dots n$ L of water. It is allowed to double the amount of water in any barrel taking it from any other barrel which contains enough water. What is the largest quantity of water which could be collected in one barrel?

a) $n = 10$

b) n is arbitrary

Solution. We have $K = n(n+1)/2$ L.

(i) Let K is odd. Then in one barrel one can collect no more than $K - 1$ L (otherwise before the last step we had $K/2$ L in each of two barrels).

(ii) Let K is even. Then in one barrel one can collect no more than $K - 2$ L. Really, we cannot collect $K - 1$ (see (i)). Assume we collected K. Consider some step reversed: we divide water in some barrel into two halves and pour one half in some other barrel. As a result each the barrel should contain $mK \cdot 2^{-s}$ L. However, we cannot represent 1 in such way because K contains odd factor.

Let us prove that these estimates are sharp. Consider two barrels containing a and b L. Pouring water between them we get $2a$ and $2b$ modulo $(a + b)$, then $2^2 a$ and $2^2 b$ modulo $(a + b)$, and so on. If $a = 2^m$ with $m \geq 1$ and b odd, we get in the first barrel after k-steps 2^{k+m} (modulo $(a + b)$). For some k $2^{k+m} \equiv 1 \pmod{a + b}$ (Euler theorem; because $a + b$ is odd). Then we can get 2 and $a + b - 2$ L of water working with these two barrel only. Then, using 2L barrel, we can pump in from some third barrel 2L making $a = 4$. Then we get the previous situation with $a + b$ increased by 2. Applying this procedure until $a + b = K$ (for odd K) or $a + b = K - 1$ (for even K), we collect K or $K - 1$ L respectively in two barrels.

Euler theorem. For natural numbers r, n: $r^{\Phi(r,n)} \equiv r$ modulo n where $\Phi(r, n)$ is Euler function.

7. The incircle of a triangle divides the segment connecting a vertex with some point on the opposite side into three equal parts. Is it possible for this segment to be the

a) altitude

b) median

c) bisector

of the triangle?

Sketch (a) no, (b) no, (c) yes.

(a), (b) Let denote triangle and segment as ABC and AK. Then $AB = BK$.

(c) Triangle with sides 5, 10, 13 with median drawn to the side equal 10.

8. Construct triangle if the lengths a and b of two sides are given and if the median drawn to the third side divides the angle in ratio

a) 1 : 2.

b) 1 : 3.

Sketch. We have equations

a) $a \sin \alpha = b \sin 2\alpha$ $(\cos \alpha = \frac{a}{2b})$; we can find $\alpha \in (0, \frac{\pi}{3})$ iff $b < a < 2b$.

b) $a \sin \alpha = b \sin 3\alpha$ $(\cos 2\alpha = \frac{a-b}{2b})$; we can find $\alpha \in (0, \frac{\pi}{4})$ iff $(\sqrt{2} + 1)b < a < 3b$.

Note that solution exists conditionally.