

MathBattle Superevent: Solutions

February 3, 2002

1. Choose units of time and distance to be equal to the time it takes Jane to make a step, and the length of her step, respectively. Let T be the time it takes Joe to make a step, and L the length of his step. Then Joe's speed equals (L/T) . We are required to compare L with 1. It follows from assumptions that $400(L/T) + 100L = 400 - 100T$. Therefore Hence $L = f(T) = T(4-T)/(4+T)$ with $0 < T < 4$. Obviously, $f(T) \leq 4/(4+T) < 1$ and therefore $L < 1$.

Answer: Jane's makes larger steps.

Remark: One can easily find the maximum of $f(T)$ and prove that $L \leq 4(3 - 2\sqrt{2}) < 0.7$.

2. Magician and Assistant from the very beginning paired cards: for example $1-2, 3-4, \dots$ etc). Then among 40 cards is at least one pair. Magician gives the pair to Spectator. So assistant picks up the "unpaired" card.
3. First player wins

Proof: Any $n > 1$ can be uniquely written as $2^l m + 1$ with odd m . We say that $n \in L$ if $l = 2k + 1$ and $n \in W$ if $l = 2k$. Any even $n \in W$ ($k = 0$).

If $n \in W$ one can transform it to $n' \in L$. Really, if $k > 0$ than using operation $n \mapsto n - 1$ one gets $2(2^{2k-1}m - 1) + 1 \in L$. If $k=0$, by operation $n \mapsto n - 1$ we get m , which belongs to L unless $m = 2^{2s}m' + 1$ with $s > 0$; in this case $n = 2^{2s}m' + 1$ and using operation $n \mapsto \lfloor \frac{n+1}{2} \rfloor$ we get $n' = 2^{2s-1}m' + 1 \in L$.

On the other hand, if $n \in L$ then n' is always in W . Really, $n - 1$ is even and thus belongs to W and $\lfloor \frac{n+1}{2} \rfloor = 2^{2k} + 1 \in W$.

So: the strategy of the first player is to produce numbers belonging to L . Then on each step the second player always produces numbers belonging to W . Since a number decays and one can produce 1 only from $2 \in W$, the first player wins.

Remark: There is a simpler implicit proof; implicit means that the existence of the winning strategy proven but this strategy is not found.

4. Let D be an intersection of AC and a perpendicular dropped from B to AL . Then triangle BKD is equilateral. Let E be an intersection of KD and AB . Then triangle EKN is also equilateral. Notice that BC is the bisector of KBD , so it is a median and an altitude as well. Therefore triangle KCD is isosceles as well, implying that $\angle CKD = \angle CDK$.

Now we have $\angle EBK = \angle KDN$ ($\triangle EBK = \triangle NKD$) and it is equal to $\angle DKC = \angle EKM$. So, $\angle EMK = \angle MKB + \angle MBK = \angle EKB = 180^\circ - 60^\circ = 120^\circ$. Because $\angle ENK = 60^\circ$, quadrilateral $EMKN$ is cyclic. Therefore $\angle NMK = 60^\circ$.

5. Denote the Lady's assigned seat by L , and the Last Passenger's by assigned seat by P . When every passenger except the last one is seated, there is one vacant seat left. It may be either L or P because each of the other seats has been occupied by then either by its legal owner, or it had been caught earlier by someone else).

By symmetry, $P(L \text{ is vacant}) = P(P \text{ is vacant}) = \frac{1}{2}$. There is indeed a symmetry between L and P , certainly for the Lady (all the seats are equal for her), and for $n-2$ other passengers as well (each of them only distinguishes between his/her legal seat, and all the other seats; the latter include both L and P).

Answer: $\frac{1}{2}$.

Remark: One can produce many nice variations of the original problem

6. Since $x^3 \equiv x \pmod{3}$ for any x , we get that $p + q = (p - q)^3 \equiv p - q \pmod{3}$. Then $2q \equiv 0$ and $q \equiv 0 \pmod{3}$. Since q is prime we conclude that $q = 3$.

On the other hand $3 \equiv (-3)^3 \pmod{p}$ and 30 is divisible by p . since p is prime, $p = 2, 3, 5$ and only $p = 5$ satisfies the equation.

Answer: $p = 5, q = 3$.

7. Draw a regular pentagon. Continuing its sides we get 5 straight lines. These lines divide the plane into several domains, of which 5 are unbounded quadrangles. Each of them has a symmetry axis, hence a circle can be inscribed into it. These five circles have the required property.
8. First, for each MP A let $W(A)$ be the set of all MPs acquaint with A (we include A into $W(A)$ as well); let $N(A)$ be a number of elements in $W(A)$. Then $W(A) \neq W(B)$ for $A \neq B$ because some third MP belongs to only one of these two sets.

Further, note that if President orders to all MPs not acquaint with A to change their affiliations then any B with $N(B) \geq N(A)$, $B \neq A$ changes his affiliation. Really, if $N(B) > N(A)$ then B should be acquaint with some MP who is not acquaint with A . It is true for $N(B) = N(A)$ as well since $W(B) \neq W(A)$.

Finally Order MPs in such a way that $N(A_1) \geq N(A_2) \geq \dots \geq N(A_n)$. If A_1, \dots, A_k belong to the same party and A_{k+1} belongs to another party, then President orders to o all MPs not acquaint with A_{k+1} to change their affiliations. Then A_1, \dots, A_k but not A_{k+1} will change their affiliations and A_1, \dots, A_{k+1} will belong to the same party.

By induction, President can make all MPs to belong to the same party. If it is a wrong party, President orders to all MPs to change their affiliations.