✠ MATHBATTLE 4 ✠

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1. A point A belongs to the interior of a circle and does not coincide with its center O. Find all the points M of the circumference of the circle such that $\angle AMO$ is maximal.

2. On the circumference of a circle there are n points numbered by $1, 2, 3, \ldots, n$ in the clockwise direction. Two players are playing the following game: in turns each player draws a chord connecting two points of the same parity. It is not allowed to intersect the chords drawn already (even at the endpoints). The person who cannot make a move loses. Who has a winning strategy (for each $n = 4, 5, 6, \ldots$)?

3. There are five coins identical in appearance but of different weights. For any three of them, say A, B, C, it is allowed to ask a yes-or-no question whether the statement "$m(A) < m(B) < m(C)$" is true. Is it possible to order the coins according to their weight if one may ask no more than nine questions?

4. Prove that for non-negative numbers x,y,z such that $x^2 + y^2 + z^2 = 1$ the following inequality holds:
$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} \geq \frac{3\sqrt{3}}{2}.$$

5. What is the minimal number of weights such that it is possible to divide them into 4, into 5, and into 6 piles of equal mass?

6. Points D and E are chosen on the lateral sides AB and AC of the isosceles $\triangle ABC$. It is known that $AD = BC = EC$ and that the $\triangle ADE$ is isosceles.
Find all the possible values of angle A.

7. In the country Absurdia there is an odd number of air companies. The first company serves just one line, the second company serves two lines and so forth. Each line connects two cities. According to the Law of Absurdia no company can serve more than one line from the same city. But one day it was decided that each company should serve the same number of lines. Can this be arranged in accordance with the Law?

8. The numbers $2, 3, \ldots, n+1$ and all the possible products of pairs, triples, etc. of these numbers, up to the product of all n of them, are written on the blackboard. Find the sum of the reciprocals of these numbers.

9. The sum of n numbers is equal to 0. Let m and M be the minimal and the maximal of them. Prove that
(a) the sum of the squares of these numbers does not exceed $-mMn$;
(b) the sum of the fourth powers of them does not exceed $-mMn(m^2 + M^2 + mM)$.

10. There are 1000 trenches in a row. A soldier is hiding in one of them. Each turn a cannon shoots at any particular trench and kills the soldier if he is there. After each shot the soldier (if alive) runs to an adjacent trench. Is there a strategy for the gunner which ensures the kill after some specific (possibly large) number of shots?