MathBattle 5: Season 2002/2003: Problems

June 22 , 2003

1. There are 2003 coins on the table. Two players play at the following game: in turns they pick up coins. First Player may pick up any odd number of coins from 1 to 99, while Second player may pick up any even number of coins from 2 to 100. Player, who can not make the move, loses and should hand his coins over to his adversary. 1) Who has the winning strategy? 2) How many coins he can get?

2. During geometry class Peter drew a trapezoid on a black board. He constructed the midline and the perpendicular from point of intersection of diagonals to the larger base. During the brake, Mary erased everything, but both constructed segments. Is is possible to restore initial trapezoid?

3. Point K is chosen on bisector AL of $\triangle ABC$ in such a way that $\angle BKL = \angle KBL = 30°$. Straight lines AB and CK intersect at point M while AC and BK at point N. Find $\angle AMN$.

4. 100 prisoners served life sentences in The Jail. Warden announced that from tomorrow all of them will be completely isolated from one another. From time to time one of prisoners will be brought in the room with a single lamp which he can either turn on or turn off. Eventually, every prisoner would visit room many times. At the moment when one of the prisoners announces correctly that every prisoner has visited this room, all of them would be set free. However, if the announcement is wrong, all of them would be executed. Could the prisoners work out a strategy which would set them free?

(a) If it is known that initially the lamp is turned off.

(b) If the state of lamp is unknown.

5. Each square of the chess board is occupied by a couple of cockroaches. At some moment each cockroach moved to an adjacent (by a side) square and all the couples were separated. What is the maximal possiblen number of cockroach-free squares?

6. Let $\star$ be a binary operation, defined on $\mathbb{R}$

(a) satisfying the following condition

$$\forall a, b, c \in \mathbb{R} \qquad (a \star b) \star c = a + b + c.$$

(b) satisfying for some N the following condition

$$\forall a_1, a_2, \ldots, a_N \in \mathbb{R} \qquad (\ldots((a_1 \star a_2) \star a_3) \ldots) \star a_N = a_1 + a_2 + \cdots + a_N.$$

Could be $a \star b$ be different from $a + b$?

7. Points P and Q are located on sides AB and AC of $\triangle ABC$. Let T be the point of intersection of segments CP and BQ. Find locations of P and Q to maximize the area of $\triangle PQT$.

8. The Island is inhabited by Liars and Truthtellers. One day each inhabitant made two statements:

(a) All people I know are acquainted between themselves.

(b) I know at least as many Liars as Truthtellers.

Find the bounds for ratio of Liars and Truthtellers.

9. Solve equation:

$$\left\{(x+1)^3\right\} = x^3, \qquad \{x\} = x - \lfloor x \rfloor.$$

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1. One can prove that if the number of coins is a multiple of 101, then FP loses (for any number of coins FP picks up, SP controls the number of coins after his move $N \equiv 0 \mod 101$).

Otherwise, FP wins. Perfect strategy: note, $2003 = 19 \cdot 101 + 84$. In his first move FP takes 81 coin leaving $19 \cdot 101 + 3$ coins on the table. On each his next move he complements the number of coins taken by SP to 101. After each next move of FP the number of coins left is $18 \cdot 101 + 3$, $17 \cdot 101 + 3$, ..., 3. Then SP is forced to take 2 cons, leaving the last one for FP to take and win.

2. Yes, it is possible if it is known which segment is midline. Let midline be KL, perpendicular OM and Q their intersection. Denote $KQ = p$, $QL = q$, $OQ = h_0$, $QM = h$, $h > h_0$. Let KL be horizontal with K the left-most , QM vertical with M the lowest. First, let us find bases a,b of trapezoid. Similarity of triangles implies that

$$a = (p+q)\frac{h-h_0}{h}, \qquad b = (p+q)\frac{h+h_0}{h}.$$

Further, let N be the point symmetrical to M with respect to KL. Let $x = AN$, where A is the top-left vertex, x could be negative. From similarity of triangles we find

$$x = \frac{(h_0 q - hp)(h - h_0)}{2hh_0}.$$

3. Let us consider point D symmetrical to B with respect to extension AL. Then $KD = KB$ and $\angle BKD = 60°$. Therefore, $\triangle BKD$ is equilateral. Then BC is bisector. So K and D are symmetrical with respect to BC. Let us consider point E which is intersection of AB and KD. Then E and N are symmetrical with respect to AL. So $\triangle NKE$ is equilateral. Then $\angle ENK = 60°$. Further

$$\angle KBM = \angle KDC = \angle DKC = \angle MKE,$$
$$\angle EMK = \angle MBK + \angle BKM = \angle BKE = 120°.$$

So, quadrilateral $EMNK$ is cyclic. Then $\angle EMN = \angle EKN = 60°$.

4. a) They elect Smart Guy. Each prisoner (save SG) turns the lamp on once (first time when he finds it turned off); he ignores the lamp otherwise. SG turns the lamp off every time it is on, and counts. On count 99 he announces.

b) In this case each prisoner (save SG) turns the lamp on twice (first and second time when he finds it turned off); he ignores the lamp otherwise. SG turns the lamp off every time it is on, and counts. On count 199 he announces.

5. (i) There can be no more than 24 cockroach-free squares.

Let us consider black squares a1, a5, c3, e1, b8, d6, f4, h2, f8, h6. One can check easily that they do not have common adjacent white squares. Therefore 20 cocroaches originally sitting in these black squares will occupy 20 white squares and therefore no more than 12 white squares will be free. Similarly, there will be no more than 12 free black squares.

Example (exactly 24 squares): 4 central squares (all cockroaches go outwards) and the "frame" with corners a2,a7,g7,g2 (corner roaches go outwards, all others - one outwards, one inwards). All others are staying in their original "frames".

6. (a) Equality

$$(a \star b) \star c = a + b + c \tag{1}$$

implies that $(0 \star 0) \star c = c$ $(a = b = 0)$. Denoting $\phi = 0 \star 0$ we get $\phi \star c = c$ for any c. Then $(\phi \star \phi) \star \phi = \phi$. On the other hand, from (1) $(\phi \star \phi) \star \phi = 3\phi$; therefore $\phi = 0$. Then we have $(0 \star b) \star c = b \star c$ and from (1) $(0 \star b) \star c = b + c$. So, $b \star c = b + c$ for all b, c.

(b) Plugging $a_1 = \dots = a_{N-1} = 0$ into equation

$$\forall a_1, a_2, \dots, a_N \in \mathbb{R} \qquad f_N(a_1, \dots, a_n) \stackrel{\text{Def}}{=} (\dots((a_1 \star a_2) \star a_3) \dots) \star a_N = a_1 + a_2 + \dots + a_N. \tag{2}$$

we conclude that

$$\forall a \quad \phi \star a = a, \qquad \phi = \underbrace{(\dots((0 \star 0) \star 0) \dots) \star 0}_{N-1 \text{ times}}. \tag{3}$$

Then plugging $a_1 = \dots = a_N = \phi$ into (2) we get that $f_N(\phi, \dots, \phi) = n\phi$ while (3) implies that $f_N(\phi, \dots, \phi) = \phi$. Then $\phi = 0$.

Plugging $a_1 = \dots = a_{N-2} = 0$ into (2) we get $f_N(0, \dots, a, b) = a + b$ while (3) and $\phi = 0$ imply that $f_N(0, \dots, a, b) = a \star b$.

7. We prove that the area of $\triangle PQT$ is maximal if

$$\frac{AP}{AB} = \frac{AQ}{AC} = \frac{\sqrt{5} - 1}{2}.$$

Let $\vec{AB} = \vec{b}$, $\vec{AC} = \vec{c}$. Then $\vec{AP} = p\vec{b}$, $\vec{AQ} = q\vec{c}$ with $p, q \in (0, 1)$. Then $\vec{PC} = \vec{c} - p\vec{b}$ and $\vec{QB} = \vec{b} - q\vec{c}$.

Further, $\vec{BT} = n\vec{BQ}$, $\vec{PT} = m\vec{PC}$ with $m, n \in (0, 1)$. Since $\vec{BP} + \vec{PT} = \vec{BT}$ we conclude

$$(p - 1)\vec{b} + m\big(\vec{c} - p\vec{b}\big) = n\big(q\vec{c} - \vec{b}\big)$$

or $(p - 1 - pm + n)\vec{b} + (m - qn)\vec{c} = 0$ and therefore $(p - 1 - pm + n) = (m - qn) = 0$ which implies that

$$m = \frac{q(1 - p)}{1 - pq}, \qquad n = \frac{(1 - p)}{1 - pq}.$$

Then

$$\vec{TP} = \frac{q(1 - p)}{1 - pq}\big(p\vec{b} - \vec{c}\big), \qquad \vec{TQ} = \frac{p(1 - p)}{1 - pq}\big(q\vec{c} - \vec{b}\big).$$

So,

$$S_{\triangle PQT} = \frac{1}{2}\big|\vec{TP} \times \vec{TQ}\big| = \frac{1}{2} f(p, q)\big|\vec{b} \times \vec{c}\big|$$

with

$$f(p, q) = \frac{pq(1 - p - q + pq)}{1 - pq}.$$

Since $p + q \geq 2\sqrt{pq}$, we conclude that

$$f(p, q) \leq \frac{pq(1 - 2\sqrt{pq} + pq)}{1 - pq} = \frac{pq(1 - \sqrt{pq})}{(1 + \sqrt{pq})}$$

with equality only for $p = q$. Denoting $\sqrt{pq} = x$ we see that $f(p, q) \leq \phi(x) = \dfrac{x^2(1 - x)}{1 + x}$. Differentiating, we find that $\phi(x)$ reaches its maximum on $(0, 1)$ as $x^2 + x - 1 = 0$ or $x = \dfrac{\sqrt{5} - 1}{2}$. Thus, $f(p, q)$ reaches its maximum as $p = q = \dfrac{\sqrt{5} - 1}{2}$.

8. Ratio Truthtellers : Liars $\in [1, \infty[$.

(i) Let us prove that the number of Liars l does not exceed the number of Truthtellers t. Let us appoint one of Liars Chief-Liar L_1; let G_1 be a group of people he knows. Let us appoint one of Liars, not familiar with L_1, Chief-Liar L_2, form the group G_2 and so on until we have no more Liars left. Note that Chief-Liars do not know one another. Let t_k, l_k be the numbers of Truthtellers and Liars (including Chief-Liar) in the group G_k. Then $t_k \geq l_k$ (otherwise L_k would tell the truth) and therefore $\sum_k t_k \geq \sum_k l_k \geq l$. On the other hand, each Truthteller belongs to no more than one group G_k and therefore $t \geq \sum_k t_k$. Thus $t \geq l$.

(ii) It can be that $t = l$. Really, let there be l Liars and l Truthtellers; each Liar knows everyone and each Truthteller knows all Liars.

(iii) There can be very many Truthtellers: really, we can always add hermit-Truthteller.

9. Obviously $0 \leq x < 1$. It follows from $\left\{x^3 + 3x^2 + 3x + 1\right\} = x^3$ that $n = 3x^2 + 3x$ is an integer with $n = 0, 1, \ldots, 5$. Then $x = \frac{-3+\sqrt{9+12n}}{6}$.