MathBattle 4: Season 2002/2003: Problems

March 9 , 2003

1. Joe keeps all his money in a Loonie Bank. He has \$500 on his account. Bank allows only two operations: a withdrawal of \$300 or a deposit of \$198. Find the maximal amount Joe can withdraw (in a few steps).

2. P is a point inside of equilateral $\triangle ABC$. Consider A' and C', points of intersection straight lines AP and BC, and CP and BA, correspondingly. Find the locus of point P, for which $AA' = CC'$.

3. In a Chess Tournament each player played with each other player twice (once as white and another as black). It happened that the final scores of all players were equal (win = 1, draw = 0.5, loss = 0). Is it always true that there are two participants who playing as white won the same number of games?

4. Electric circuit contains 2000 contacts. Each pair of contacts is connected by a wire. Joe and Bill are playing the following game: in turns they remove wires. Joe starts and on his turn removes one wire while Bill removes either two or three wires. Who cuts the last wire from one of the contacts, loses. Who has a winning strategy?

5. There are 26 coins in a collection, all of different weights. There is a list of weights of coins. Collection supervisor claims that he knows the weight of every coin. He wants to prove it by using scales which show the difference in weights between cups.

Find the minimal number of weighings he needs.

6. An iceberg in the shape of a convex polyhedron is floating in the ocean. Could it happen that no less than 90% of its volume is below while no less than 50% of its surface is above sea level?

7. Numbers 1, 2,..., 2003 are written on a blackboard (in order).

John and Mary in turns erase some of the numbers: John erases all the numbers on odd places; then Mary erases all the numbers on even places.

They continue until one number is left. What is this number?

8. What is the minimal number of paper triangles needed to cover a cube (without overlapping)?

9. The skier made a round-trip visiting each of n villages exactly twice. Is it always possible to make a trip following his trail but visiting each village only once? (village = point, trail = line).

MathBattle 4: Season 2002/2003: Solutions

1. Since both 300 and 198 are multiple of 6, Joe can withdraw only multiple of 6. The largest multiple of 6 not exceeding 500 is 498. Joe can withdraw this amount:

$$500 - 300(= 200) + 198(= 398) - 300(= 98) + 198(= 296) + 198(= 494)$$

Notice that amount decreased by 6. Repeating this procedure 15 times more Joe withdraws 96 (never being below 0) and has 404. Finally

$$404 - 300(= 104) + 198(= 302) - 300(= 2).$$

2. Lets choose any point A' on BC and consider AA'. There are only two segments CC' and CC'' ($C'' \in AB$, $AC'' = BC' = BA'$) equal to AA'. Point of intersection of AA' and CC' belongs to altitude BH of triangle ABC; all points of BH satisfy the condition.

Let P be the point of intersection AA' and CC''. Let us notice that triangles ACC'' and BAA' are equal, so $\angle APC = 180° - (\angle PAC + \angle PCA) = 180° - \angle CAB = 120°$. That means that P is on circumference passing through A, C and intersection of altitudes of triangle ABC. All points of this circumference inside of triangle ABC satisfy the condition.

3. The total number of games is $n(n-1)$ where n is the number of players. Therefore the final score of each player is $(n-1)$. Each player played $(n-1)$ games as white. Assume that all players won different number of games as white. Then reordering players if necessary, we see that k-th player won $(k-1)$ games as white. Consider player A who won $(n-1)$ game as white and player B who won 0 games as white. Consider the game A (bl.) - B (wh). In this game A got at least 0.5 points (because B did not win it) and he also took $(n-1)$ points in games he played as white. So, A score is at least $n - 0.5$. Contradiction.

4. Bill wins. Let us divide contacts into 4 equal groups A, B, C, D and number contacts in each group by $1, \dots, 500$. Let us mark wires connecting contacts with equal numbers.

Bill's strategy is maintain some kind of symmetry: namely, he keeps the numbers of unmarked wires, connected to contacts A_k, B_k, C_k, D_k (with the same k) equal. Let us consider three possible Joe moves:

(1) Joe removes a wire connecting contacts in the same group, say $A_i - A_j$. Bill responds by removing wires $B_i - B_j$, $C_i - C_j$, $D_i - D_j$.

(2) Joe removes an unmarked wire connecting contacts in different groups, say $A_i - B_j$ $(i \neq j)$. Bill responds by removing wires $A_j - B_i$, $C_i - D_j$ and $C_j - D_i$.

(3) Joe removes a marked wire, say $A_i - B_i$. Initially there are 6 wires connecting contacts A_i, B_i, C_i, D_i. If $A_i - B_i$ is the first to be removed, Bill responds removing $A_i - C_i$, $A_i - D_i$ (leaving $A_i - D_i$, $B_i - D_i$, $C_i - D_i$).

If later Joe removes one of these wires, say $A_i - D_i$, Bill responds by removing the other two. Really, if wire $A_i - D_i$ is not the last one attached to A_i (D_i) then remains either wire, say, $A_i - A_j$ or wire, say, $A_i - B_j$. In the former case wires $B_i - B_j$, $C_i - C_j$, $D_i - D_j$ are still intact; in the latter case wires $A_j - B_i$, $C_i - D_j$, $C_j - D_i$ are intact. In both cases no wire attached to B_i, C_i, D_i is the last one (attached to it).

5. Add a weightless coin to make the number of coins equal 27. Supervisor divides coins into three groups according to their weights. Then he places 9 heaviest coins on one side of the balance and 9 lightest on the other. Checking that the difference is maximal possible Supervisor proves that his division is correct. Now he marks coins according to their group: H, M, L.

Then he takes 3 heaviest coins from each group and weights these 9 coins against 9 coins (3 lightest coins from each group). The difference again is maximal possible in this arrangement. Supervisor marks every coin by a second letter H, M, L. So now he has 9 groups marked by $HH, \dots, LL$.

Finally he tests 9 heaviest coins (one from each group) against 9 lightest coins (one from each group) and again the difference is maximal possible. Then he marks coins by a third letter and these 3 letters show place of each coin in the ranking.

Two weighings is not enough: there will be two coins with the same two letters markings.

6. Yes. Consider an upside-down regular pyramid with the 1×1 square base and height h. Then $9/10$ of h should be submerged, and the underwater part of surface is $2(9/10)^2 l$ while the total surface is $2l + 1$ where l is a height of a lateral face. So we need $2(9/10)^2 l \leq l + \frac{1}{2}$ or $62l < 50$. This is possible since we can take any $l > 0.5$.

7. After the first round (John erases, Mary erases) the numbers left are $2, 6, \dots, 2002$ ($4k - 2$ with $k = 1, \dots, 501$).

Since Mary cannot erase the last number left, we can assume that all rounds were complete. The last remaining number occupied place 2 prior to this round.

Going back, this number was on places 6, 22, 86, 342, 1366. We must stop now. Answer: 1366

8. First let us notice that no vertex can be covered by an interior of triangle. So, it should be covered by edges. Note that when an interior of edge covers a vertex, the sum of adjacent angles covered by triangle is exactly 180°. At the same time the sum of angles adjacent to vertex of cube is 270°. Therefore at least 90° at each vertex should be covered by angles of triangles. So angles of triangles cover at least $8 \cdot 90^\circ$ and there should be at least $8 \cdot 90^\circ/180^\circ$ of triangles.

Consider T-shaped envelope of cube, consisting of two rectangles. Each of them can be covered by 2 triangles. So, it is possible to cover a cube by 4 triangles.

9. Not necessary. Consider round-trip

$$A \to B \to E \to E \to B \to C \to F \to F \to C \to A \to D \to D \to A.$$

If there is a trip visiting each village D, E, F only once, then one of them, say D should be in the middle of the trip. However in this case it is impossible to visit A only once unless A is starting/end-point of the trip. But then either E or F (say E) is in the middle of the trip and then B should be an starting/end-point. Then F and C are in the middle of the trip and C is visited at least twice.