MathBattle 3: Season 2002/2003: Problems

Feb 2, 2003

1. Segment OA is split into white and black sub-segments so that the total lengths of white and black parts are the same. For any black sub-segment consider the product of its length on the distance from O to its midpoint; now let $\mathcal{B}$ be the sum of all such products. For any white segment consider the product of its length on the distance from A to its midpoint; let $\mathcal{W}$ be the sum of all such products. Prove that $\mathcal{B} = \mathcal{W}$.

2. Point B_1 is given on side AC of $\triangle ABC$. Construct points C_1 and A_1 on sides AB and BC correspondingly so that the areas of $\triangle AB_1C_1$, $\triangle A_1BC_1$ and $\triangle A_1B_1C$ are equal.

3. First Player places numbers $1, 2, 3, \ldots, 2003$ into cells of 1×2003 table according to his choice; then Second Player places a checker into the cell of his choice. Then, in turns, they move the checker to neighbouring cell. On any cell the checker can be placed no more times than the number written on it. The player who cannot make a move, looses. Who has a winning strategy?

4. A 90×90 hall is divided into cubicles, each cubicle being 10×10. There is a door on each wall between the neighbouring cubicles; but there are no doors on the exterior wall. Find the maximal number of doors, that can be opened, given that each room can have no more than

a) one open door.

b) three open doors.

5. There is a field, a straight road, and a man at point O on the road. He can move along the road at 6 km/h or go through the field at 3 km/h. Describe the set of points, where he can be in an hour.

6. A point inside of an equilateral triangle is connected with vertices and the perpendiculars are dropped to each side of the triangle from it, so that the triangle is split into six triangles, blacks and whites, in alternating order. Prove, that total sums of radii of circles inscribed into black and white triangles are the same.

7. 1000 aboriginals, each of them being either a Liar or a Truthteller, live on an island. The New Governor came for a short visit to find out the number of Liars. Of course, everyone who lives on the island knows who is who, but the Governor only knows that not all of aboriginals are Liars. Every day he may choose any group of aboriginals and ask everyone in this group how many Liars are among them. Find the minimum number of days that are needed to find out the number of liars on the island.

8. Find the minimal length of the sum of 2003 unit vectors with nonnegative coordinates in 3-dimensional space.

MathBattle 3: Season 2002/2003: Solutions

1. Let us construct parallelogram $OPAQ$ such that its sides PO and QA are equal and perpendicular to OA. Each term of $\mathcal{B}$ equals the area of black trapezoid, constructed above the corresponding black segment and each term of $\mathcal{W}$ equals the area of white trapezoid, constructed under the corresponding white segment.

Now, let us cut parallelogram $POQA$ into vertical stripes, each containing either a black or a white trapezoid. Lets call $\mathcal{B}$-figure a combination of all stripes, containing black trapezoids. Area of $\mathcal{B}$-figure is equal to the half of the area of $POQA$.

Area of $\triangle OPA$ (consisting of white trapezoids plus $\mathcal{B}$-figure minus black trapezoids) is also the half of $POQA$. It means that $\mathcal{B} = \mathcal{W}$.

2. Let us construct series of straight lines: through B_1 parallel to AB (intersecting BC at M), through M parallel to AC (intersecting AB at C_1), through C_1 parallel to BC (intersecting AC at K), through K parallel to AB (intersecting BC at A_1). It is easy to see that $\triangle AB_1C_1$, $\triangle A_1BC_1$ and $\triangle A_1B_1C$ have equal areas (equal the half of "equalareas" parallelograms).

3. SP wins.

Next to every cell of the table let us put a column of checkers, the number of checkers equals the number written in the cell. Now, let us pair checkers in columns. We start from the first column. If the number of checkers in the first column is greater than the one in the second, then some checkers in the first column would left without pair; if less, then some checkers of second column have no pairs. In the latter case let us pair leftovers with checkers of third column. Again, either some checkers of the second column would be left without pair or all of them would be paired. In the latter case we proceed with leftovers of the next column and etc. When process is over, at least one checker is not paired (the total number of checkers is odd).

Now, consider the first checker which is left unpaired (the top in the leftmost column). SP places checker on the cell, corresponding to this column. No matter where FP moves, SP always returns to cell he started.

4. Let us color cubicles in black and white as a chessboard with blacks at the corners. Then we have 40 whites and 41 blacks; every door connects cubicles of different color. So, the total numbers of open doors in black and white cubicles are the same.

a) The answer is 40. Really, the number of open doors in white cubicles does not exceed 40; it could be achieved:

Example 1. Denote cubicles by $(m, n) : m, n = 1, \ldots, 9$. Open doors between cubicles $(2k-1, n)$ and $(2k, n)$ with $k = 1, \ldots, 4$, $n = 1, \ldots, 9$ and also between cubicles $(9, 2l-1)$ and $(9, 2l)$ with $l = 1, \ldots, 4$. All other doors are closed.

b) the answer is 119. First, the number of open doors does not exceed 120. However, 4 black corners have exactly $2 \times 4 = 8$ doors. So, the total number of open doors does not not exceed $8 + 37 \times 3 = 119$. It could be achieved:

Example 2. Close doors between cubicles $(2k, n)$ and $(2k+1, n)$ with $k = 1, \ldots, 3$, $n = 2, \ldots, 8$ and also between cubicles $(8, 2l)$ and $(8, 2l+1)$ with $l = 1, \ldots, 4$. All other doors are open.

5. Introduce coordinate system with OX as a road. Consider the first quadrant. Let man walk to $(z,0)$ $(0 \le z \le 6)$ along the road and then walk through the field. At the end of 1 hour he can be anywhere in the disk $D(z, 3-\frac{z}{2})$ of radius $3-\frac{z}{2}$ with the center at $(z,0)$. One can prove that the envelope of these disks is a straight line $y = (6-x)\tan 30° = (6-x)/\sqrt{3}$. However, this line touches $D(0,3)$ at point $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$. Therefore the region in question (in the first quadrant) is bounded by segment $\{y = \frac{1}{\sqrt{3}}(6-x), \frac{3}{2} \le x \le 6\}$ and a circular arc $\{y = \sqrt{9-x^2}, 0 \le x \le \frac{3}{2}\}$.

6. Proof follows from two propositions.

Proposition 1. *Diameter of circle, inscribed into the right angle triangle equals the sum of its two legs minus hypotenuse.*

Proposition 2. *Sums of perimeters of black and white triangles are the same.*

Proof. Let P be a given point. Through P draw lines parallel to sides of the triangle. The statement follows from cancelling out equal segments of different colors.

7. Show that two days is enough. The first day Governor invites all aboriginals. Receiving their answers, he includes people, who gave the same answer in the same group. Let the number of groups be n. All Truthtellers are in one group. All Liars would be in the remaining $n-1$ groups. If $n = 1$, then all aboriginals are Truthtellers). If $n > 1$, then on the second day Governor invites just one representative from each group. Among chosen would be one Truthteller, who gives the answer "$n-1$". So he and all his groupmates of the first day are Truthtellers.

One day is not enough. Assume that 1 aboriginal claims that there are 999 Liars and all others claim that there is 1 Liar. So the number of Liars could be either 1 or 999.

8. Let $\mathbf{s}$ be a fixed sum of any n vectors, and $\mathbf{a}$ be a variable unit vector. Then $\|\mathbf{s}+\mathbf{a}\|$ is minimal if $\mathbf{a}$ creates with $\mathbf{s}$ the maximal angle. One can prove that it happens when $\mathbf{a}$ is one of three coordinate unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

So, the 2003 vectors split into 3 groups containing k, m, n vectors; $k+n+m = 2003$.

The square of the sum is $k^2+m^2+n^2$ and it reaches its minimum 1337337 as $\{k,l,m\} = \{667, 668, 668\}$.

So, the answer is $\sqrt{1337337}$.