MathBattle 2: Season 2002/2003: Problems

Jan 12, 2003

1. There are 49770 mobsters. For each of them the total number of his immediate bosses and immediate subordinates is equal to 7. On Monday every mobster issued an instruction and distributed copies of it to all his immediate subordinates (if there were any). On Tuesday each mobster who received instruction delivered copies of it to his immediate subordinates. The same went on Wednesday and Thursday . It is known that was no distribution on Friday. Find the minimal number of supreme bosses (who have no bosses).

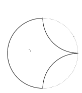
2. Council of Wizards is tested in the following way: the King lines all the wizards up in a line and places either a white, blue or red hat on each of them. Every wizard can see the color of hats of the wizards in front of him, but cannot see the color of his hat or hats of the wizards standing behind. According to rules, every minute some of the wizards must name one of the three colors. Also, one is allowed to speak out just once. When testing is over, the King executes all the wizards who failed to name correctly the color of their hats. Prior to this procedure, wizards had agreed to minimize the number of executions. How many of them are definitely secure?

3. There 333 isles in Lake Ontario. Inspector Gadget on one of them and Dr. Claw on the other. Every day at noon from each island boats depart to some other isles and arrive to their destination at 11 am next day. It is known that from any island one can reach any other island. Gadget always knows the location of Dr. Claw, but he does not know which boat Dr. Claw could take. Also, Gadget takes no days off but Dr. Claw never takes boat on Sunday. Can Gadget catch Dr. Claw for sure?

4. Among 24 coins there are two counterfeits (one is heavier and the other is lighter than a real coin). Using a simple balance find the minimal number of weighings which is needed to determine if the total weight of two counterfeits equals the total weight of two real coins.

5. Points M and N are chosen inside the square $ABCD$ so that $\angle MAN = \angle MCN = 45°$. Prove that the sum of the areas of triangles ABM, AND, MNC is equal to to the sum of the areas of triangles AMN, BNC, CND.

Remark: originally it was "equal to a half of the area of $ABCD$".

6. A figure  is bounded by a semi-circle and two quarter-circles of the same radius. Cut it into five pieces so that one of them is a square and four others could be put together (without holes and overlaps) to form a congruent square.

7. There are three empty jars on a table. Winnie the Pooh, Rabbit and Piglet take turns putting nuts into the jars (one at the time; each has a sufficient number of nuts). According to rules, Winnie the Pooh can put a nut into the first or second, Rabbit into the second or third, and Piglet into the first or third jars. The order of players had been defined randomly, prior to the game. The player looses if a jar contains 2003 nuts after his move. Can Winnie the Pooh and Piglet working together force Rabbit to lose?

8. Prove that for non-negative a, b, c

$$\frac{(a+b+c)^2}{3} \geq a\sqrt{bc} + b\sqrt{ac} + c\sqrt{ba}.$$

MathBattle 2: Season 2002/2003: Solutions

1. Obviously, there are at most five levels in the chain of command.

a) Consider the maximal M, the number of mobsters under supervision of 1 top level boss. He has exactly 7 immediate subordinates. To maximize M, each mobster of levels 2,3,4 should have exactly one immediate boss (and therefore exactly 6 immediate subordinates). So, the maximal numbers of mobsters on levels 3 and 4 are $7 \cdot 6$ and $7 \cdot 6 \cdot 6$ respectively. However, each mobster of level 5 has no subordinates, so should be under supervision of 7 immediate bosses. Therefore, there is $7 \cdot 6 \cdot 6 \cdot 6 : 7$ mobsters of level 5. Thus, the total is $M + 1 = 512$ and 96 bosses is not enough (42 would be a remainder).

b) Let us construct an example with 97 bosses; structure for 96 bosses is given in (a) and 97-th boss leads a family of 42 (himself included). We put 7, 18, 14,2 mobsters on levels 2-5. One can prove that this scheme is possible in frames of conditions.

2. We can save all but the last (N-th) wizard. Let us assign numbers 0,1,2 to white, blue and red colors of hats respectively. Each wizard calculates S_i (mod 3) the sum of colors of all wizards ahead of him (where i is his number in the line). Then the N-th wizard names S_N (as a color of his hat). After this $(N-1)$-th wizard calculates $a_{N-1} = S_N - S_{N-1}$ (mod 3) and names it as his color. Then every wizard calculates $S_{N-1} = S_N - a_{N-1}$ and $(N-2)$-th wizard names $a_{N-2} = S_{N-1} - S_{N-2}$ (mod 3) as his color and so on....

3. Let the distance from island A to island B be the minimal number of days one can reach B from A. When C. moves he increases the distance from G. to him by at most 1. When G. follows the shortest possible way to C. he decreases this difference by 1. So, each ordinary day the distance does not increase while once in a week it necessarily decreases.

4. *Answer:* 3 weighing. Let us divide all coins into four groups of six (A, B, C, D). Two first weighings are: A vs B and C vs D.

a) $A = B$, $C = D$. The answer is YES.

b) $A > B$, $C > D$ (etc). Third weighing: $A + D$ vs $B + C$. If $A + D = B + C$ the answer is YES, otherwise the answer is NO.

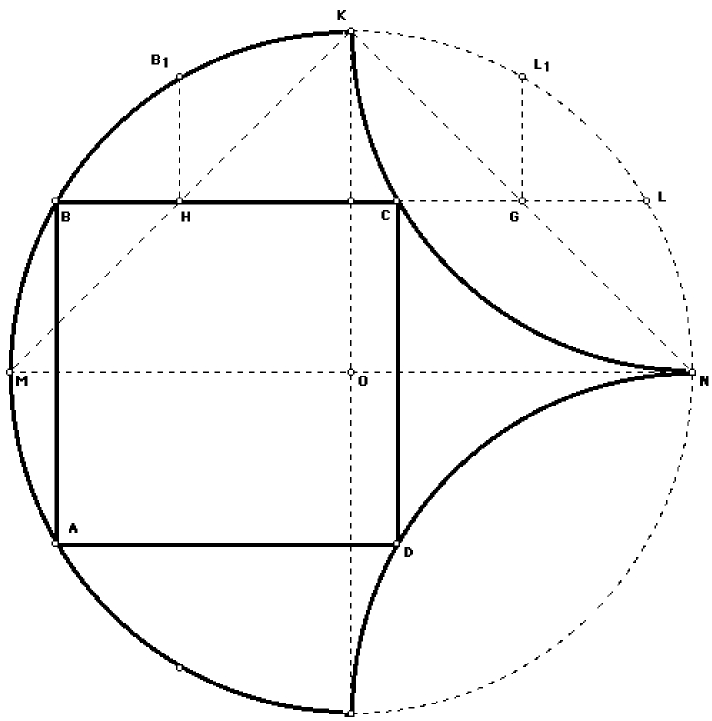
c) $A = B$, $C \neq D$ (etc). Third weighing: $A + B$ vs $C + D$. If $A + B = C + D$ the answer is YES, otherwise the answer is NO.

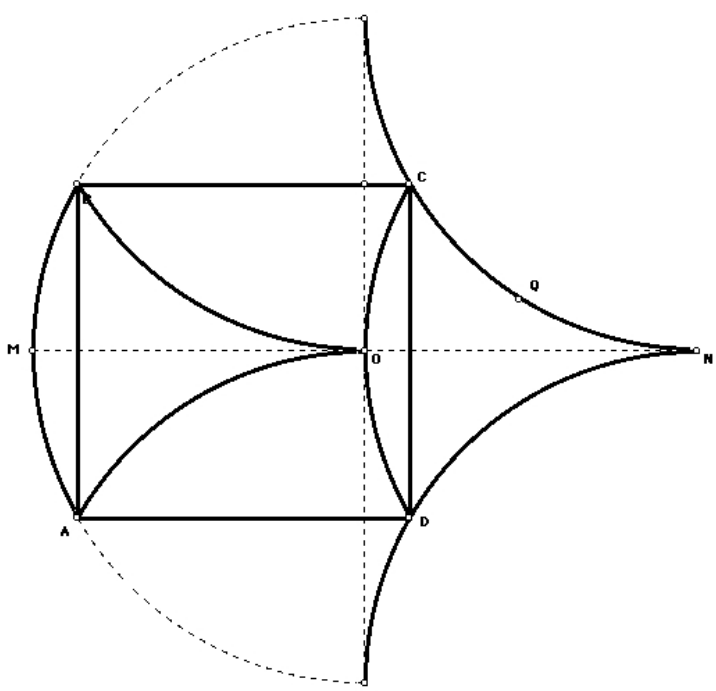
One can see that 2 weighings are not enough.

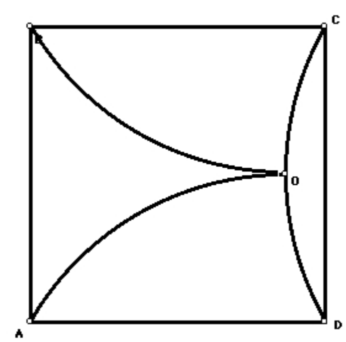
5. Let us reflect AB with respect to AM and AD with respect to AN. Then reflections of points B and D coincide; denote it by K; $AB = AD$ and $\angle BAM + \angle AND = 45°$. So, the sum of the areas of triangles ABM, AND, MNC is is equal to the area of quadrilateral $AMCN$ plus an area of $\triangle MNK$.

Let us reflect BC and CD with respect to MC and NC respectively and let L be a common image of B and D; then the total area of the rest of the square is equal to the area of quadrilateral $AMCN$ plus an area of $\triangle MNL$. Note that $\triangle MNK$ and $\triangle MNL$ are equal (S-S-S) which completes the proof.

6. Let the radius be 1, then the area of the figure is 2 and each square should have side 1. Let us inscribe rectangle $ABCD$ into figure with $AB = CD = 1$. Let us prove that it is a square. HG is a midline of $\triangle MKN$ and so $HG = 1$. Note that $BH = B_1H$ ($B_1H \perp BH$). Due to symmetry $BH = GL = GL_1$, therefore $BC = HG = 1$. The rest is clear from the following pictures.







7. W. places his nuts into the second jar (ignoring moves of R.) until it contains 2002 nuts. Then he switches to the first jar. P. uses the same strategy with the third jar.

Let W. be the first to switch. To this moment W. already placed into second jar at least 1001 nut and therefore P. placed in the third jar at least 1001 nut. So, P. continues to fill the third jar, and R. is forced to fill the third jar, and they will done with in no more than 500 move. In that moment P. switches to the first jar contains no more than 501 nuts and R. is forced to loose.

8.

$$a\sqrt{bc}+b\sqrt{ac}+c\sqrt{ba}\le a\cdot\frac{b+c}{2}+b\cdot\frac{a+c}{2}+c\cdot\frac{b+a}{2}=ab+ac+bc\le$$
$$\frac{2}{3}(ab+ac+bc)+\frac{1}{3}(a^2+b^2+c^2)=\frac{1}{3}(a+b+c)^2.$$