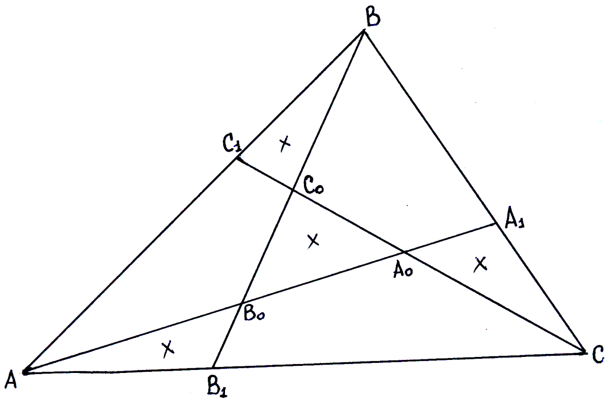
MathBattle 1: Season 2002/2003: Problems

Nov 24, 2002

1. It is given that four marked triangles have the same area.

Prove that so have the three quadrilaterals $A_0C_0BA_1$, $A_0CB_1B_0$ and $AC_1C_0B_0$.



2. Jeremy solves quadratic equations. Solving the equation $x^2 + p_1x + q_1 = 0$ he checks that it has two real roots p_2 and q_2, $p_2 < q_2$. Then he solves equation $x^2 + p_2x + q_2 = 0$ and so on.... How long can this exercise continue if Jeremy does not know complex numbers?

3. Construct a quadrilateral (with a compass and a straightedge) if the lengths of its four consecutive sides and the length of the segment connecting the midpoints of its diagonals are given.

4. Five couples met on the Mr. Brown's party. Some of the people shook hands (naturally nobody shook hands with their own spouse). Mr. Brown asked everyone about the number of hands he/she shook and got 9 different numbers. How many hands did Mrs. Brown shake?

5. Sequence a_n is defined by the conditions $a_1 = 0$, $a_{2n+1} = a_{2n} = n - a_n$.

(a) Find a_{2002}.

(b) Calculate $a_n - a_{n-1}$ for any n.

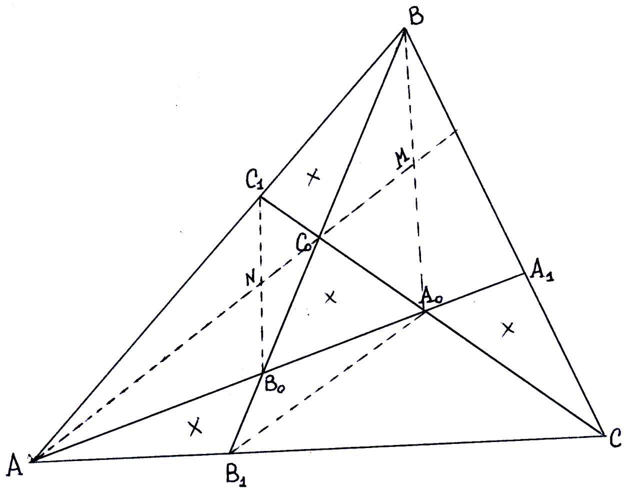
6. In one village live 1000 people; among them are 300 Truthtellers (who always tell the truth) and 700 Normals (who sometimes tell the truth and sometimes lie). You may ask any number of people any number of questions. What number of Normals can you identify for sure?

7. A Mathematician and a Physicist have a piece of cheese (2001 × 2002 × 2003 - block) . First Physicist divides the block by a straight cut into two smaller blocks (with integer sides) and eats one of them; then Mathematician does the same thing and so on. Whoever eats a $1x1x1$-block first wins. Who has the winning strategy?

8. Inspector Gadget tries to catch Dr. Claw in a Y-shaped dungeon which contains three straight corridors OA, OB and OC of the same length L connected at point O and A, B, C are dead ends. It is known that Gadget moves twice as fast Dr. Claw and that Gadget can notice Dr. Claw at a distance not exceeding R. To start, Gadget is at O and does not see Dr Claw. Can Gadget catch Dr Claw (and how should he act)? Consider cases:

(a) $R = L/3$; (b) $R > L/5$; (c) $R > L/7$.

1. See notations below.



It is sufficient to prove that $C_0B = C_0B_1$ (and $AB_0 = B_0A_1$ which is done in the same way). Really, in, this case $S(\triangle C_0BC) = S(\triangle B_1C_0C)$; therefore $S(A_1A_0C_0B) = S(CA_0B_0B_1)$, where $S(K)$ means "area of K".

Let us prove that $B_0C_1BA_0$ is trapezoid. Indeed, $S(\triangle A_0B_0B) = S(\triangle A_0C_1B)$ and therefore B_0 and C_1 are equidistant from A_0B and then $B_0C_1 \parallel A_0B$. So, straight line passing through A and C_0 divides B_0C_1 and A_0B into halves: $A_0M = MB$. In the same way we can prove that $A_0C_0AB_1$ is trapezoid as well with $AC_0 \parallel B_1A_0$. Then C_0M is a midline of $\triangle B_1BA_0$ (because $C_0M \parallel B_1A_0$ and $A_0M = MB$). So, $C_0B = C_0B_1$.

2. Let $f_n(x) = x^2 + p_nx + q_n = 0$ be n-th equation in a sequence.

Proposition. *If $n \geq 3$ and $f_n(x) = 0$ is not the last equation, then $p_nq_n > 0$.*

Proof. By Vietta theorem,

$$p_{n-1} = -(p_n + q_n), \qquad q_{n-1} = p_nq_n. \tag{1}$$

Further, $p_n < q_n$ for any n. Plugging $n-1$ instead of n and taking into consideration (1), we get that $-(p_n + q_n) < p_nq_n$. So:

$q_n > p_n$, $q_n > -p_n(1 + q_n)$, $4q_n < p_n^2$ (since equation is not the last). The statement follows easily.

Let us assume that the sequence contains at least 5 equations. If $f_5(x) = 0$ is not the last equation, then by Proposition we have that <u>either</u> $0 < p_5 < q_5$ and then $p_4 < 0 < q_4$ <u>or</u> $p_5 < q_5 < 0$ and then $0 < p_4 < q_4$ and then $p_3 < 0 < q_3$. Both cases contradict to our proposition.

So, the sequence contains at most 5 equations.

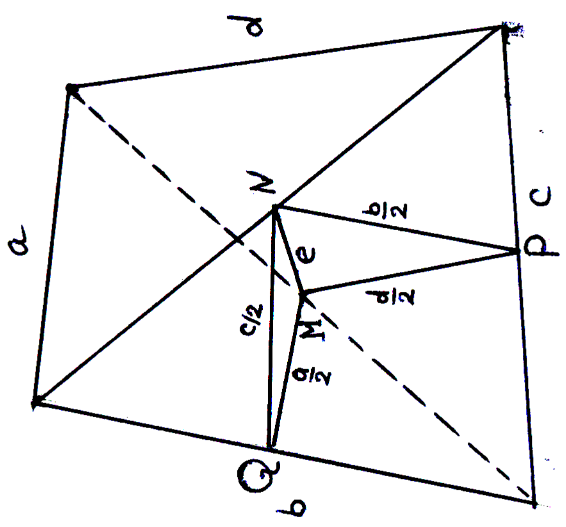
The following example shows that the sequence of 5 equations exists: $f_5(x) = x^2 - \frac{1}{2}x + 4$, $f_4(x) = x^2 - \frac{7}{2}x - 2$, $f_3(x) = x^2 + \frac{11}{2}x + 7$, $f_2(x) = x^2 - \frac{25}{2}x - \frac{77}{2}$, $f_1(x) = x^2 - 26x - \frac{1925}{4}$.

3. Let a, b, c, d and e be given lengths of consecutive sides and the segment, connecting midpoints M and N of diagonals.

Analysis. Let us connect M and N with midpoints P and Q of sides c and b. From the property of midlines of triange it follows that $MP = d/2$, $NP = b/2$, $NQ = c/2$, $MQ = a/2$ and that segments MP, NP, NQ, MQ are parallel to the corresponding sides. Problem has a solution iff $\max(|d-b|, |a-c|) < 2e < \min(d+b, a+c)$.

Construction. Construct triangles $\triangle MNP$ with sides $MP = d/2$, $NP = b/2$ and $MN = e$ and $\triangle MNQ$ with sides $NQ = c/2$, $MQ = a/2$ and $MN = e$. Then, draw line through P parallel to MQ and through Q parallel to NP. Intersection of them will be the vertex between

b and c, two adjacent vertices are recovered easily, and then the last vertex is constructed easily as well.



4. Consider respondent with a maximal number of handshakes (Eight). Mrs Brown is not Eight because otherwise every guest shook hands with her and no respondent had 0 handshakes. Since Eight shook hands with everybody but the spouse, only Eight's spouse could be Zero. So, Eight and Zero are spouses. Similarly, then Seven and One must be spouses, Six and Two, Five and Three. Thus Mrs Brown is Four: she shook hands with Five, Six, Seven, Eight (and Mr Brown did the same).

5. Let us decompose $n = 2^k s$ where s is an odd number. *Then* $a_n - a_{n-1} = 1$ *if* k *is odd and* $a_n - a_{n-1} = 0$ *if* k *is even.*

If $k = 0$, then $a_n = a_{n-1}$ by definition.

Consider $n = 2^k s$ with $k \geq 1$. Then $a_n = \frac{n}{2} - a_{n/2}$ and $a_{n-1} = \frac{n-2}{2} - a_{(n-2)/2}$ and therefore $a_n - a_{n-1} = 1 - (a_m - a_{m-1})$ with $m = \frac{n}{2} = 2^{k-1}s$. Then for $b_k = a_{2^k s} - a_{2^k s-1}$ we have $b_k = 1 - b_{k-1}$ and therefore $b_0 = 0$, $b_1 = 1$, $b_2 = 0$, $b_3 = 1, \ldots$.

6. We can identify 100 Normals for sure. We ask everyone to classify everybody including himself. If A classifies as Truthtellers not 300 people, or does not include in this number himself, then he is Normal. If one of 300 people classified by A as Truthtellers by A contradicts him, then A is Normal. Therefore, all Truthtellers must belong to a *closed* group of 300 (we call group closed if every member of this group classifies everybody in this group as Truthteller, and everybody else as Normal). There can be no more than 3 closed groups (1 is a real Truthteller group). This leaves at least 100 people. On the other hand, there can be exactly 3 closed groups which cannot be classified. In this case we can identify exactly 100 Normals.

7. The first player (P) is a winner. Really, if M leaves $1 \times n \times m$ with $n < m$ then P leaves $1 \times n \times n$ block. Eventually M is forced to leave $1 \times 1 \times m$ block with $m > 1$. So, in his first move P leaves $2 \times 2001 \times 2002$. Then not to follow the previous scenario, M cannot divide "2". Neither can he leave $2 \times 2001 \times 2001$ or $2 \times 2 \times 2001$ (otherwise P will leave $1 \times n \times n$ block). Let M leave $2 \times p \times q$ block where $q = 2r$ or $p = 2r - 1$ ($r = 2, \ldots, 1000$). Then P leaves $2 \times (2t - 1) \times 2t$-block. P continues this way and eventually M gets $2 \times 3 \times 4$-block. The rest is easy to check... Well, M agreed to these conditions because he noticed that cheese was not exactly fresh.

8. First, G. investigates corridor OC completely and returns to O.

(a) $R = L/3$: G. goes $2R$ into OA (and finds that C. is not there) and return to O. During this time C cannot pass from OB to OC (otherwise G. returning to O could see him). Then G. goes to OA.

(b) $R > L/5$. G. goes $y_1 = 2R$ into OA, and returns to O. If C. was in OA, he can be on the distance no less than $y_1/2 + R$ from O If C. was in OB, he could not escape into OC. First step is done. Then G. goes into OB on the distance $y_2 = y_1/2 + 2R$. And so on, G. keeps switching corridors OA and OB, pushing C. further and further: $y_{n+1} = y_n/2 + 2R$. Then $y_n = 4R(1 - 2^{-n})$. G. eventually catches C. provided $y_n + R \geq L$ which is the case for large n iff $5R > L$.

(c) $R > L/7$ First G. follows plan (b) until y_n is so close to $4R$ that $L - y_n < 3R$. Then G. goes into OB until he sees its end. If C. is not there, G. returns to O. Now C. is either in OA or he slipped into OC but in this latter case his distance from O is at most z_0 with $z_0 < 3R$. Then G. travels into OC at the distance $2z_0 - 2R$. If C. is there, G. will see him. If C. is not there G. returns to O. Now C. is either in OA or he slipped into OB but in this latter case his distance from O is at most $z_1 = 2z_0 - 3R$. Now G. goes into OB at the distance $2z_1 - 2R$ and so on: G. keeps switching corridors thus "pulling" C. to O: $z_{n+1} = 2z_n - 3R$. Then $z_n = 3R - 2^n(3R - z_0)$. After few steps $z_n \leq R$ and G. (who is at this moment in O) either sees C. or knows which corridor C. is in.

(d) Remark. There was proven that is no sure method to catch C. if $L \geq 7R$.