MathBattle 10: Problems & Solution.s

May 19, 2002

1. Planar convex figure is bounded by a circular arc DB and two segments AB and AD.

Construct a straight line which divides into halves

a) perimeter of the figure;

b) area of the figure;

Solution. Let M be a midpoint of the arc.

(a) Let P be a point on DA (if $DA > DB$) or DB (otherwise) such that $AP = PD + DB$ (or similarly). Then MP be a required straight line.

(b) Note that areas of a figure bounded by arc AM and segment AM is equal to the area of a figure bounded by arc BM and segment BM and therefore we need to find a straight line passing through M and dividing quadrilateral $DAMB$ into two figures of equal areas. Let N be a midpoint of segment AB; consider a straight line passing through N and parallel to MD; this straight line meets DA or DB at some point Q and MQ is a required straight line.

2. Let a, b, c be real numbers such that $(a + b + c)c < 0$. Prove that $b^2 > 4ac$.

Solution. Consider quadratic polynomial $P(x) = ax^2+bx+c$. Since $P(0) = c$, $P(1) = a+b+c$ have different signs, discriminant should be positive.

3. There is a circular hole somewhere in a square-shaped golf field. A grass-hoper hops on this field: each time it chooses a vertex and hops in its direction covering exactly the half of the distance. Can he reach the hole?

Solution. This problem should be solved backwards. Consider our hole which is a disk $D(O_0, r_0)$ with the center at O_0 and radius r_0. Let A_0 be a vertex of the field which is closest to O_0. Grass=hoper will reach $D(O_0, r_0)$ by a jump towards A_0 iff it was in $D(O_1, r_1)$ with O_1 such that $\vec{A_0O_1} = 2\vec{A_0O_0}$ and $r_1 = 2r_0$. Obviously O_1 belongs to our field. In the same way we select A_1 and construct O_2 and $r_2 = 2r_1$ and so on. Finally we get a disk $D(O_n, r_n)$ covering the whole field. Now if grass-hoper jumps towards A_{n-1}, then A_{n-2}, then A_{n-3} etc, it will reach $D(O_0, r_0)$.

Question. What happens if grass-hoper covers in its jump 1/3 of the distance to the chosen vertex? 2/3?

4. *Graph* is a set of *vertices*; some of them are connected by *edges*. Each edge connects exactly two vertices. We call the coloring of the vertices (graph) *regular* if no two vertices of the same color are connected by an edge. Let us consider a graph regularly colored into k colors such that it cannot be regularly colored into $k-1$ colors. Prove that there exists a path containing k vertices, all of them of different colors.

Solution. Let us consider the regular k-coloring. Then for any two colors, say i and j, there exists a pair of connected vertices of these colors. Really, otherwise repainting i into j we would have a regular $k - 1$-coloring.

Now, consider all the vertices of color 2 and repaint those of them which are not connected with vertices of color 1, into color 1. Coloring remains regular and each vertex of the color 2 will be connected to some vertex of color 1. Then we repaint vertices of color 3, not connected with vertices of color 2, into color 2, and so on, until color k. Note that coloring remains regular.

Mark all the vertices which preserved their colors. Then each marked vertex of color $j \geq 2$ is connected to some marked vertex of color $j-1$.

Now, there still will be a vertex of color k, because there is no regular $(k-1)$-coloring. This vertex will be connected to some marked vertex of color $k-1$, connected in turn to some marked vertex of color $k-2$ and so on, until the marked vertex of color 1. We have our chain!

5. Seven cyclists, each starting at different times, rode in one direction. One of them had a waterbottle. From time to time, one cyclist got ahead of another. If one of them had a bottle he gave it to the other one. The bottle was not transfered in any other way. What is the minimal number of times when somebody got ahead of someone else if every cyclist had the bottle at least once?

Solution. Let us number cyclists according to their order at the initial moment $1, \ldots, 7$ and according to their position at the current moment $(\bar{1}, \ldots, \bar{7})$. If the bottle was at the initial moment with cyclist k, it will be at any moment with cyclist $\bar{k}$. Therefore for cyclist $j > k$ to reach the bottle, he must advance forward at least $j-k$-times, and for cyclist with number $j < k$ reach the bottle, he must advance backwards at least $k-j$ times. So the minimal number of overcomes in "before the bottle" group is $1+2+\ldots+(k-1) = k(k-1)/2$ and the minimal number of overcomes in "after the bottle" group is $1+2+\ldots+(7-k) = (7-k)(8-k)/2$ and the sum is $(k(k-1)+(7-k)(8-k))/2$. This number reaches its minimum as $k=4$ and this minimum is 12. One can easily show that 12 overcomes is enough.

6. There is a frame obtained from a rectangular $m \times n$ $(m \geq 3, n \geq 3)$ board by removing all interior squares. Two players cut it in turns. In one move it is allowed to cut out any number of squares in row from the same side of the frame provided that the remaining part does not split into two pieces. The player who makes the last move wins. Who has a winning strategy?

Solution. The first has a winning strategy. Really, let him cut the corner square. The second player should cut a piece from one of adjacent sides. If it was a winning answer, the first player would be able to cut the corner square+this piece in his first move and then follow the winning strategy of the second player.

6*. Two coiners play the following game: in turns they issue a new coin of integer value. It is not allowed to issue a coin of value 1 or a coin already issued or a coin which could be exchanged by any number of coins already issued. The coiner who cannot issue a new coin loses.

a) Prove that the game cannot continue infinitely.

b) Who has a winning strategy?

Solution. (a) We will use the following

Lemma 1 *If $a_1, \dots a_n$ are positive integers then there exists m such that every integer $z \geq m$ multiple of $d = \gcd(a_1, \dots, a_n)$ can be represented as $z = a_1x_1 + \dots a_nx_n$ with non-negative integers $x_1, \dots, x_n$.*

Let $a_1, \dots, a_n$ be coins issued at moment n and let $d_n = \gcd(a_1, \dots, a_n)$. Note that $d_1 \geq d_2 \geq d_3 \dots$. Also note, that for given $a_1, \dots, a_n$ there exists m such that every coin multiple to d_n and larger than m can be exchanged by $a_1, \dots, a_n$ and thus is forbidden. So, each value of d_n cannot be repeated infinitely. Thus the game cannot continue indefinitely.

(b) The first has the winning strategy. The first coiner starts from 5 (or any prime $p \geq 5$. The second must answer by $q > 1$ with $\gcd(p, q) = 1$.

Lemma 2 *The largest integer z which cannot be represented as*

$$z = px + qy \tag{1}$$

with non-negative integers x, y is $m = pq - p - q$. Further, for $z \leq m$ exactly one of the numbers z, $m - z$ can be represented as (1) .

Proof We can represent z as (1) with $0 \leq x \leq q-1$. If $y < 0$ then $m - z = p(q-1-x) + q(-y-1)$ is represented as (1) with non-negative coefficients. Inversely, if $y \geq 0$ then $-y-1 < 0$. Since $z = 0$ is the smallest number which admits representation, then $m - 0$ is the largest which does not.

So, after the move of the second coiner the largest allowed value is $m = pq - p - q$. Let the first coiner 's second move be m. If it is loosing strategy, then the second coiner has a winning response $z, 1 < z < m$. But then the first coiner could play z as his second move and then follow the winning strategy of the first one. Really, according to Lemma 2, $m - z = px + qy$ with non-negative x, y and number $m = z + pz + qy$ will be forbidden after this move of the first coiner, and the sets of forbidden numbers after moves p, q, z and after moves p, q, m, z are the same.

7. A set (possibly with repetitions) of 2001 numbers satisfies the following condition: if every number is replaced by the sum of all other numbers, then the set does not change. Find the product of all numbers in the set.

Solution. If each number a is replaced by $S - a$ where S is the sum of all the numbers, then the new sum will be $2000S$; from assumption it follows then that $S = 0$. Therefore, each number and its opposite enter the set the same number of times; since 2001 is odd, set must contain 0 and the product is equal to 0.

8. 49 buttons are arranged in 7×7 square. Every button can be either ON or OFF. Pressing each button we change the states of all buttons in 3×3 square with the center at this button. Is it possible to have all 49 buttons off simultaneously?

Solution. First, any 3×1 rectangle we can switch off all the buttons pressing only buttons of this rectangle.

Really, let its buttons be A, B, C. We switch off A and C by pressing them if needed, then switch off B by pressing it if needed. Then both A and C are either off, or on. In the second case we press A,C switching them off and leaving B off.

Second, let us consider 3×3 square with columns A, B, C and prove that one can switch off all buttons in it (pressing only them). First we do it for columns A and C; then we will do it for column B; we could spoil A and C but notice that states of the buttons of A and C are symmetrical with respect to B. Finally we switch off A and C and since each button of B changes its state even number of times, we get B off again.

Third, consider 3×7 rectangle consisting of squares A, B and 3×1 rectangle B in the middle. We switch off A and C, then B and then A and C again as above.

Also similar arguments work for 1×7 rectangle.

Finally, we repeat the above arguments for upper and lower 3×7 rectangles A,C and 1×7 rectangle B in the middle.

9. Find a seven digit number such that its first digit equals to a number of "0" in it, its second digit equals to a number of "1" in it, and so on,

Solution. Note that the sum of the digits of the number should be equal to 7. Then by trial and error one can find the number 3211000 and prove that it is unique:.