International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Spring 2015**

Problem 1. Pete summed up 100 consecutive powers of 2, starting from some power, while Basil summed up several consecutive positive integers starting from 1. Can they get the same result?

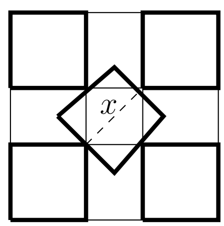
Answer. Yes, they can.

Solution. Indeed, let $2^{k+1} + \dots 2^{k+100} = 1 + 2 + \dots + n$. Simplifying we see that $2^{k+2}(2^{100} - 1) = n(n + 1)$ holds for $n = 2^{100} - 1$ and $k = 98$. □

Problem 2. A moth made four small holes in a square carpet with a 275 cm side. Can one always cut out a square piece with a 1 m side without holes? (Consider holes as points).

Answer. One can always cut out a square piece with a 1 m side without holes.

Solution. On the picture one can see the positions of 5 non overlapping squares, four corner squares with side of 1 m and a central square with side $x = 0.75\sqrt{2} > 1$. Four moths can make holes in at most four of these pieces.



□

Problem 3. Among $2n + 1$ positive integers there is exactly one 0, while each of the numbers $1, 2, ..., n$ is presented exactly twice. For which n can one line up these numbers so that for any $m = 1, \dots, n$ there are exactly m numbers between two m's?

Answer. For any n.

Solution. Observe that two sets of odd numbers, each set from from 1 to $2k + 1$ can be arranged according to the requirement with one empty space in the middle:

$$2k + 1, 2k - 1, \dots, 3, 1, \boxed{}, 1, 3, \dots, 2k - 1, 2k + 1$$

while two sets of even from from 1 to $2k$ can be arranged according to the requirement with two empty spaces in the middle:

$$2k, 2k - 2, \dots, 2, 1, \boxed{}\boxed{}, 1, 2, \dots, 2k - 2, 2k$$

(a) $n = 2k+1$. Consider the follow arrangement:

$$2k{+}1, 2k{-}1, \dots, 3, 1, \boxed{2k}, 1, 3, \dots, 2k{-}1, 2k{+}1, 2k{-}2, 2k{-}4 \dots 2, \boxed{2k, 0}, 2, \dots, 2k{-}2$$

Inserting two copies of $2k$ as shown, we see that for any $m \neq 2k$ requirement holds and we can check that it holds for $m = 2k$ as well.

Indeed,

$$1, 3, \dots, 2k-1, 2k+1, 2k-2, 2k-4 \dots 2,$$

includes $k+1$ of odd numbers and $k-1$ of even numbers, $2k$ numbers in total.

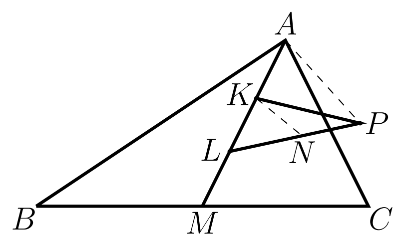
(b) $n = 2k$. In a similar way one can check that the following arrangement works:

$$2k-1, 2k-3, \dots, 3, 1, \boxed{2k}, 1, 3, \dots, 2k-1, 2k-2, 2k-4 \dots 2, \boxed{0, 2k}, 2, \dots, 2k-2$$

. $\square$

Problem 4. Points K and L are marked on the median AM of triangle ABC, so that $AK = KL = LM$. Point P is chosen so that triangles KPL and ABC are similar $\left(\frac{KP}{AB} = \frac{PL}{BC} = \frac{KL}{AC}\right)$. Given that points P and C are on the same side of line AM, prove that point P lies on line AC.

Solution. Let KN be median of triangle LKP. Since triangles ABC and KPL are similar, triangles LKN and CAM are similar as well. If $\angle CAM = \alpha$, then $\angle LKN = \alpha$. On the other hand, triangles LKN and LAP are similar ($KN \parallel AP$ as a midline of triangle LAP). Therefore, $\angle LAP = \angle LKN = \alpha$. Hence $\angle LAC = \angle LAP$ and therefore P lies on the line AC.



$\square$

Problem 5. 2015 positive integers are arranged in a circular order. The difference between any two adjacent numbers coincides with their greatest common divisor. Determine the maximal value of N which divides the product of all 2015 numbers, regardless of their choice.

Answer. $N = 3 \cdot 2^{1009}$.

Solution. Observe that in order to satisfy the assumptions the sequence of numbers cannot contain two adjacent odd numbers. Therefore it contains at least 1008 even numbers and the product of the numbers is divisible by 2^{1008}. Assume that the product of the numbers is not divisible by 2^{1009} (so that no even number is divisible by 4). By PHP there are two adjacent even numbers. Then their difference is divisible by 4. Therefore one of these even numbers must be divisible by 4. Contradiction.

Let us prove that the product of the numbers is divisible by 3. Assume that no number is divisible by 3 (so it equals 1 or 2 modulo 3.).

Moreover, reminders 1 and 2 must alternate (otherwise the difference is divisible by 3 and one of the numbers is divisible by 3). Since the total number of alternating objects must be even we got a contradiction.

Hence the product of the numbers is divisible by at least 3×2^{1009}.

Example that 3×2^{1009} is reached:

$$\underbrace{2,\ 1, 2,\ \ldots,\ 2,\ 1}_{1006\ \text{“2”},\ 1006\ \text{“1”}}, 2, 3, 4$$

$\square$